1 Recap

1. Examples of stability and instability.

2 Stability

More examples:

1. Consider the Lorenz system (originally for atmospheric convection but later for simplified models of lasers, chemical reactions, etc - the original butterfly effect): $x' = \sigma(-x+y), y' = Rx - y - -xz, z' = -bz + xy$. Assume all constants are positive. If $R \leq 1$, the origin is the only equilibrium but for $R > 1$, $(\pm\sqrt{b(R-1)}, \pm\sqrt{b(R-1)}, R-1)$ are two more points. For $R < 1$, the linearisation at the origin has negative eigenvalues (and hence asymptotically stable even nonlinearly) and when $R > 1$, one eigenvalue is positive. When $R > 1$, the eigenvalues at the other points are roots of $\lambda^3 + (\sigma+b+1)\lambda^2 + (R+\sigma)b\lambda + 2\sigma b(R-1) = 0$. It has a real eigenvalue for sure. When $R >> 1$, the derivative is positive and $f(0) > 0$. Thus there is only real eigenvalue and it is negative. It turns out that for large R (the Routh-Hurwitz criterion derived by simply factorising the cubic), the real parts of the other eigenvalues is negative. Note that we transit from linear stability to instability at $R = 1$ at the origin. So $R = 1$ is called a bifurcation point.

2. Lastly, non-autonomous systems can be very strange: Consider $x' = A(t)x$ where $A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin^2(t) \end{bmatrix}$. It turns out that (using power series or simply guessing) a basis of linearly independent solutions is $v_1 = e^{t/2}(-\cos(t), \sin(t))$ and $v_2 = e^{-t}(\sin(t), \cos(t))$. Thus the equilibrium point $(0,0)$ is unstable and of saddle type. However, the eigenvalues of $A(t)$ are $\frac{-1 \pm \sqrt{7}i}{4}$ for all t!

3 Liapunov functions

To analyse the situation of non-hyperbolic equilibria (even for hyperbolic ones, it is not easy as we shall see), a powerful tool is the Liapunov function. Suppose 0 is an isolated equilibrium point and Ω is a neighbourhood that does not contain other equilibria (any other point can be analysed by translation).
Def: A C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, $V(x) > 0$ for $x \in \Omega - \{0\}$, $\nabla V.F \leq 0$ in Ω is called a Liapunov function.
Here is an important theorem: Suppose a Liapunov function V exists. Then the origin is stable. If $\nabla V.F < 0$ on $\Omega - \{0\}$, then 0 is asymptotically stable.
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Proof. The point is that $\frac{dV}{dt} = \nabla V.x = \nabla V.F \leq 0$ and hence V is decreasing along the trajectory. Suppose we consider a sphere C of radius ϵ (where ϵ is given but small) such that it and its interior lies entirely within Ω, and let m be the minimum of V on this sphere. For all $\|x\| \leq \delta$, $V \leq \frac{m}{2}$. If $\|x(t_0)\| \leq \delta$, then originally $V \leq \frac{m}{2}$. If at some point of time, $\|x(t)\|$ crosses C, then V would have to be $> \frac{m}{2}$ which is a contradiction. Hence $\|x(t)\|$ is bounded (and hence the maximal interval is all of $\mathbb{R}$) and bounded by the given ϵ and hence 0 is stable.
To be continued.... □