1 Recap

1. Lorenz system and a weird example of a non-autonomous system (unstable but linearly stable).
2. Suppose a Liapunov function V exists. Then the origin is stable.

2 Liapunov functions

Suppose $\nabla V.\vec{F} < 0$, then the origin is asymptotically stable.

Proof. If strict inequality holds, then V is strictly decreasing. Now suppose $L = \lim_{t\to\infty} V(x(t))$ is positive. Then consider a smaller sphere $\tilde{C}$ such $V(x) \leq \frac{L}{2}$ inside the corresponding ball. By assumption, $V(x(t)) \geq L$ along the trajectory and hence cannot intersect $\tilde{C}$. Thus $\nabla V.f \leq -k$ along the trajectory which provides a contradiction (why?)
Now we can show that $\lim_{t\to\infty} \|x(t)\| = 0$. Indeed, if there is a sequence of times $t_n \to \infty$ for which $\lim_{n\to\infty} x(t_n) = x \neq 0$, then $\lim_{n\to\infty} V(x(t_n)) = V(x) > 0$ which is a contradiction. □

Here is an instability result (Chetaev's theorem)

Theorem 1. *Suppose there is a C^1 function $V : \Omega \to \mathbb{R}$ satisfying $V(0) = 0$, and there exists a $\tilde{\epsilon} > 0$ so that in every neighbourhood of size $< \tilde{\epsilon}$ of 0, there is a non-empty set where $V > 0$ and $\nabla V.\vec{F} > 0$ on the region $V > 0$, then 0 is unstable.*

Proof. What we want to show is that there exists a sequence $(x_0)_n \to 0$ and a sequence t_n such that $\|x(t_n)\| \geq \tilde{\epsilon}$ or for infinitely many terms of the sequence, the solution does not exist for all positive time: Choose a closed ball $B_n \subset \Omega$ of size $\epsilon_n = \frac{\tilde{\epsilon}}{2n} < \tilde{\epsilon}$ centred at 0. There exists an $(x_0)_n \in Int(B)$ such that $V((x_0)_n) > 0$. Consider a solution with $x_n(0) = (x_0)_n$. We see that $t \to V(x_n(t))$ is increasing for $t \geq 0$. Thus the positive orbit is confined to the set $V > 0$. The claim is that this orbit crosses $\partial B_{\tilde{\epsilon}}$ at some finite time (note that if the orbit does not exist for all $t \geq 0$, it is not stable by definition anyway). Indeed, suppose not. Then it remains in a compact set such that $V \geq k > 0$. (Indeed, it certainly remains in a compact ball around the origin. Now $V(x_n(t)) \geq V(x_n(0)) > 0$.) Thus $\nabla V.\vec{F} \geq m > 0$ on the orbit and that is a contradiction (because it implies V can grow in an unbounded manner). □

We now consider a few examples.

1. Consider $x' = y, y' = x - x^3$. Note that $V = xy$ satisfies $\nabla V.\vec{F} = (y, x).(y, x - x^3) = y^2 + x^2 - x^4 > 0$ when $x^2 < 1$ (and $(x, y) \neq 0$). Thus by the above theorem, the origin is unstable.
2. Consider $x' = -y^3, y' = x^3$. The only equilibrium is the origin and there $Df(0) = 0$. Thus Perron is not applicable. Consider $V = x^4 + y^4$. Then $V(0) = 0$, $V > 0$ away from the origin, and $\nabla V.\vec{F} = (4x^3, 4y^3).(-y^3 + x^3) = 0$. Thus the origin is stable. Actually in this case, the orbits stay on $V = c$. Thus the origin is not asymptotically stable.