1 Recap

1. "Definition" of ODE. Examples/Counterexamples (including the "multiply by the inverse of the guessed solution" trick).
2. Examples coming from real life.

2 Linear systems of ODE

Consider $y'' = y$ on (a, b), how does one solve it? One possibility is to try $y = Ae^x + Be^{-x}$. It works but how do we know that this is the general solution? How did we guess at this solution in the first place? The usual way to do this in physics is the "energy method". Multiplying by $2y'$ on both sides, $((y')^2)' = (y^2)'$ and hence $(y')^2 - y^2 = E$ (Conservation of energy). Thus $y' = \pm\sqrt{E + y^2}$. We can now attempt to solve this using separation as usual. We observe two things:

1. The presence of a conserved quantity allowed us to express the solution as an integral. It is not a coincidence that the equation has a symmetry (time-translation). A deep theorem called Noether's theorem tells us that every symmetry gives rise to a conserved quantity. If you have "enough" number of conserved quantities, you can express the solution using integrals. (Such a system is called an integrable system.) Unfortunately, just because you write something as integral does not mean you can evaluate it using "elementary functions". Thankfully, we know when one can solve integrals. This topic is called differential Galois theory.
2. Despite the conserved quantity above, the integrals involved are rather hairy.

Another approach to this problem is as follows: Introduce a variable $v = y'$. Then $v' = y$, $v = y'$ is equivalent to the above equation. The cost is that now we have a *system* of ODE as opposed to a single one. More generally, a linear inhomogeneous non-constant coefficient/non-autonomous system is $\vec{y}' = A(t)\vec{y}(t) + \vec{F}(t)$ where $A(t)$ is a matrix-valued function and $\vec{F}(t)$ is a given function. We shall mostly deal with constant matrices A (linear inhomogeneous autonomous systems). For now, let us assume $\vec{F}(t) = 0$, i.e., $\vec{y}' = A\vec{y}$ (a linear homogeneous autonomous system). There are two ways of trying to solve these kinds of equations (that we shall illustrate with the above example):

1. Note that if the system was "decoupled" (that is, $v' = av$, $y' = bv$), we could have easily solved it. The idea is to change variables to get it to this form. Note that if this was a linear system of equations (not differential), then we would have simply added and subtracted equations. That is what we shall do. Indeed, $(v + y)' = (y + v)$, and $(v - y)' = -(v - y)$. Thus $v + y = Ae^x$ and $v - y = Be^{-x}$. Thus we are done. More generally, if $\vec{y}' = A\vec{y}$, a general change of variables is $\vec{z} = P\vec{y}$ where P is an invertible matrix. Then $\vec{z}' = PAP^{-1}\vec{z}$. This system is decoupled if $PAP^{-1} = D$ is a diagonal matrix, i.e., when A is diagonalisable! (the columns of P^{-1} are a basis of eigenvectors of A!) By the way, this is why you should care about eigenvalues and eigenvectors. Unfortunately, not everything is diagonalisable and even if it is, the eigenvalues can be complex. So stay tuned....

2. For $y' = ay$, we know the solution is $y = e^{ax}y_0$. If we can define e^{At}, then is $y = e^{At}y_0$ the solution? Indeed, $(e^{-At}y)' = -Ae^{-At}y + e^{-At}y' = 0$ and hence $y = e^{At}y_0$. We shall take up this approach too. By the way, computing e^{At} on a quantum computer efficiently is a big deal. In fact, quantum computers were originally envisaged by Feynman precisely to do this. This is called the Hamiltonian simulation problem.

2.1 Decoupling strategy

Let's pursue the first strategy. Prior to that, here is a useful easy result (HW): First, if define $z(t) = u(t) + \sqrt{-1}v(t) : (a,b) \to \mathbb{C}$ to be differentiable if u, v are so. We can then prove the product and quotient rules. Moreover, define $e^{(x+\sqrt{-1}y)t} = e^{xt}(\cos(yt) + \sqrt{-1}\sin(yt))$. Then we can prove that this function is differentiable and its derivative is $(x + \sqrt{-1}y)e^{(x+\sqrt{-1}y)t}$.

Suppose $\vec{y}' = A\vec{y}$. Firstly, the set of real differentiable functions satisfying this equation form a subspace of the vector space of differentiable functions $\vec{f} : \mathbb{R} \to \mathbb{R}^n$ (why?) The same holds for the set of complex solutions too. In fact, a complex function solves it iff its real and imaginary parts are solutions (why?). Here is a lemma.

Lemma 2.1. *The space of real solutions V is finite-dimensional iff the space of complex solutions W is so. Moreover, the real and complex dimensions coincide.*

Proof. Firstly, a space is finite-dimensional iff there exists an integer N such that every collection of $N+1$ vectors linearly dependent iff there exists a finite set of vectors such that every vector is a linear combination.
Suppose V is finite-dimensional with dimension n and a basis $u_1(t), \ldots, u_n(t)$. Then if $z(t)$ is a solution, $z(t) = \sum_i c_i u_i(t) + \sqrt{-1}\sum_i b_i u_i(t) = \sum_i (c_i + \sqrt{-1}b_i)u_i(t)$ (why?). Thus every element of W is a linear combination of $u_i(t)$ and hence W is finite-dimensional. In fact, $u_i(t)$ are complex linearly independent (why?) and hence the complex dimension is n.
Suppose W is finite-dimensional with dimension n and a basis $z_1(t), \ldots, z_n(t)$. Then the real and imaginary parts of $z_i(t)$ are solutions and hence every real solution is finite real linear combination of at most $2n$ vectors. Thus V is finite-dimensional. The previous arguments now show that its dimension is that of W. □

Now suppose P is a constant invertible matrix, then $\vec{z} = P\vec{y}$ satisfies $\vec{z}' = PAP^{-1}\vec{z}$. If A is diagonalisable, that is, if $PAP^{-1} = D$ where D is a diagonal matrix of eigenvalues (possibly complex), then $\vec{z}' = D\vec{z}$. Thus $(z_i e^{-D_{ii}t})' = 0$, i.e., $z_i = (z_i)_0 e^{D_{ii}t}$. The solutions for $\vec{z}$ form an n-dimensional complex vector space (why?) Since $\vec{y}$ is a linear combination of $\vec{z}$, $\vec{y}$ also forms an n-dimensional complex vector space. Thus the space of real solutions is also n-dimensional. We shall prove later that regardless of the diagonalisability of A or the lack thereof, this fact is still true.
Moreover, if $y(0)$ is real, then $y(t)$ is real. Indeed, there exists some complex solution $y = P^{-1}z$ where $z_i = e^{-\lambda_i t}(z_i)_0$. Now the real and imaginary parts of y are still solutions. The imaginary part has 0 initial data. From the explicit formula, we see that any solution with zero initial data is identically zero. Hence we are done.