1 Recap

1. Chetaev and examples.

2 Instability

Examples:

1. Consider $x' = -2y + yz, y' = x - xz, z' = xy$. The origin is an equilibrium. The eigenvalues of the Jacobian at the origin are $0, \pm 2i$ and hence Perron is not applicable. Consider $V = c_1x^2 + c_2y^2 + c_3z^2$. Then if $c_1 = c_3 > 0$ and $c_2 = 2c_1$, $V(x) > 0$ for $x \neq 0$ and $\nabla V.\vec{F} = 0$. Thus the orbits lie on ellipsoids (and are stable but not asymptotically stable). The Liapunov theorem is not applicable here! The reason is that the equilibria are: $(0, 0, z)$, $(0, y, 2)$, and $(x, 0, 1)$, which are not isolated!

2. (Mistake in Nandakumaran's book corrected): $x' = -2y + yz - x^3, y' = x - xz - y^3, z' = xy - z^3$. The only equilibrium is the origin (indeed compare signs of x, y). Perron is not applicable (same Jacobian as the previous example). Take $V = x^2 + 2y^2 + z^2$. Then $\nabla V.F < 0$ away from the origin. Thus the origin is asymptotically stable.

3. Consider $x' = -2y + 2yz - x^3, y' = x - xz - y^3, z' = x^2z - z^3$. The origin is an isolated equilibrium (because the other equilibria arise as zeroes of one-variable polynomials). The Jacobian matrix again has the same eigenvalues as the previous example. This time $V = x^2 + 2y^2 + z^2$ works to prove asymptotic stability because $\nabla V.\vec{F} < 0$.

4. $x' = x^2 + 2y^5, y' = xy^2$ has the origin as the only equilibrium. The linearisation is 0. Now consider $V = x^2 - y^4$ (which is 0 at the origin). Now $\nabla V.F = (2x, -4y^3).(x^2 + 2y^5, xy^2) = 2x^3 + 4xy^5 - 4xy^5 = 2x^3 > 0$ when $x > 0$. Now $V > 0$ when $x^2 > y^4 \geq 0$. Unfortunately, Chetaev does not help directly here. So we go over the proof of it. Consider (x_0, y_0) such that $V(x_0, y_0) > 0, x_0 > 0$. Now either the solution does not exist for all time for a sequence of (x_0, y_0) approaching zero and hence the origin is unstable or it does exist for all time for (x_0, y_0) lying in a neighbourhood of the origin. Assume the latter. Then $\frac{dV}{dt} > 0$ as long as $x > 0$. The first time x becomes 0, we see that $V = -y^4 < 0$ and that is a problem because $V \geq V(t = 0)$ by monotonicity until that time. Hence $x > 0$ for all time and hence $\frac{dV}{dt} > 0$. Now $x^2 \geq y^4 + V(x_0, y_0)$ and hence $\frac{dV}{dt} \geq c > 0$ and $V \to \infty$ as $t \to \infty$. Thus the origin is unstable.

3 Invariant sets and manifolds

A set $S \subset \mathbb{R}^n$ is said to be invariant under $x' = f(x)$ if for any $x_0 \in S$, $x(t) \in S \; \forall \; t \in \mathbb{R}$ and positively invariant if $x(t) \in S \; \forall \; t \in [0, \infty)$.
Sometimes these sets can possess more structure. Here are some examples.

1. $x' = x, y' = -y + x^2$. The only equilibrium is the origin. The linearisation has eigenvalues ± 1. So the stable subspace and unstable subspace each have dimension 1. We can actually explicitly solve this system to get $x = x_0 e^t, y = y_0 e^{-t} + \frac{x_0^2 e^{-t}(e^{3t}-1)}{3}$. Eliminating t, $x(y - x^2/3) = x_0(y_0 - x_0^2/3)$. These are invariant subsets of the ODE. Moreover, if $x_0 = 0$, $x = 0$ and $y \to 0$ as $t \to \infty$. This set is called a "stable manifold" (what is a manifold? Whatever it is, it is a generalisation of a "smooth regular" surface in $\mathbb{R}^3$. It is supposed to be a nonlinear generalisation of a vector space.) If $y_0 = \frac{x_0^2}{3}$, then $y = \frac{x^2}{3}$ is an invariant subset. Moreover, every point on this subset goes to ∞ as $t \to \infty$ (actually, more precisely it goes to 0 as $t \to -\infty$).

2. To be continued....