1 Recap

1. Examples of stability.
2. Invariant sets and an example.

2 Invariant sets and manifolds

1. We can take a similar example in three dimensions: $x' = -x, y' = -y + x^2, z' = z + x^2$. The linearisation at the origin (the only equilibrium) has eigenvalues $-1, -1, 1$ (so hyperbolic). Thus the stable subspace is 2-dimensional and the unstable one 1-dimensional. Upon solving, $x = x_0 e^{-t}, y = y_0 e^{-t} + x_0^2(e^{-t} - e^{-2t}), z = z_0 e^t + \frac{x_0^2 e^t (1-e^{-3t})}{3}$. Thus $\|(x,y,z)\| \to 0$ (as $t \to -\infty$) iff $x_0 = y_0 = 0$. On the other hand $\|(x,y,z)\| \to 0$ iff $z_0 + \frac{x_0^2}{3} = 0$. These sets are invariant (why?)
2. Consider a previous example: $x' = -2y + yz, y' = x - xz, z' = xy$. Then $x^2 + 2y^2 + z^2 = a^2$, $x^2 + y^2 + z = b$ are invariant sets. The origin is not hyperbolic.

We want to say roughly that under some conditions (hyperbolicity), there is a nonlinear version of the stable-unstable subspaces theorem. To state this theorem rigorously, we need to know what a manifold is. The simplest nonlinear generalisation of a subspace of $\mathbb{R}^n$ is the graph of a function. Indeed, this is what manifolds are modelled after.
Def: An n-dimensional C^r (embedded sub)manifold in $\mathbb{R}^N$ is a set $S \subset \mathbb{R}^N$ (with the induced topology) such that given any point $p \in S$, there exists a neighbourhood $p \in U \subset \mathbb{R}^N$ such that $S \cap U$ is the homeomorphic image of a C^r map $f : V \subset \mathbb{R}^n \to \mathbb{R}^N$ (where V is open) such that Df has rank-n everywhere. Here r can be ∞ in which case it is called a smooth manifold. r can also be 0 (and we drop the derivative condition) in which case it is called a topological (embedded sub)manifold.

The f's are called C^r local parametrisations (and their inverses are called 'charts'). It turns out using the inverse function theorem that in fact one can write a C^r (when $r \geq 1$) manifold locally as C^r graphs over some coordinate-axes in $\mathbb{R}^N$. Indeed, just for some practice let's prove this fact. Firstly, let's recall the inverse function theorem:
Suppose $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^r map (where Ω is open and $\infty \geq r \geq 1$) and DF_p is invertible, then there exist neighbourhoods $p \in U$ and $f(p) \in V$ such that f takes U to V homeomorphically and the inverse is C^r (it is a local C^r diffeomorphism).
Given this theorem, n rows in $Df_{f^{-1}p}$ are linearly independent. Suppose they are $i_1, \ldots, i_n$. Consider the map $T(u_1, \ldots, u_n) = (f_{i_1}, \ldots, f_{i_n})$. This map is C^r and $DT_{f^{-1}p}$ is invertible. Hence by the inverse function theorem, locally, u's are C^r functions of x_{i_k}'s. Thus the rest of the f's are C^r functions of x_{i_k}'s locally.
Here are examples and non-examples of manifolds in $\mathbb{R}^N$:

1. The graph of $y = |x|$ is not a C^1-manifold in $\mathbb{R}^2$: Indeed, if it were, then near the origin, y is a C^1 function of x (why?) and that is a contradiction.
2. To be continued...