

1 Recap

1. Energy method and examples.

2 Periodic orbits

We study $\vec{x}' = \vec{F}(\vec{x})$ in two dimensions only for periodic orbits. That is, $x' = f(x, y), y' = g(x, y)$ where f, g are C^k ($k \geq 2$) functions. We first need the notion of Poincaré index. We are interested in smooth parametrised paths in \mathbb{R}^2 , i.e., smooth maps $\gamma : [a, b] \rightarrow \mathbb{R}^2$. A piecewise smooth path is continuous, smooth away from finitely many points, and smooth from the left and right everywhere. A smooth regular path is a smooth path such that $\gamma' \neq 0 \forall t$. (Likewise piecewise regular.) A closed path is one for which $\gamma(a) = \gamma(b)$. A smooth simple closed path or a smooth Jordan curve is a closed path that is 1-1 everywhere except the endpoints. The image of a path is sometimes called a curve (but terminology is often abused and paths are also called curves).

The famous Jordan curve theorem implies that piecewise smooth Jordan curves divide the plane into two connected components (one of which is unbounded) such that the curve is the boundary of both. So we can talk about the "domain" that is "inside" the Jordan curve.

The aim is to know when periodic orbits exist and when they don't. Note that along a periodic orbit, the tangent vector "rotates" by at least 2π . This means it is perhaps important to measure the (anticlockwise) "rotation index" of a vector field along a curve. Note that the vector field of interest to us will be $\vec{v} = f\hat{i} + g\hat{j}$. Let us look at some examples.

1. Let $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$ be $\gamma(t) = (\cos(t), \sin(t))$. Consider $\vec{v} = (x, y)$. We can explicitly solve $x' = x, y' = y$ to get $(x, y) = (x_0, y_0)e^t$. Clearly there are no periodic orbits. Indeed, the picture of the vector field looks like it is "diverging away" from the origin. Now the rotation of this vector field along γ is 2π of course. (We haven't defined it rigorously but we can "see" it.) Note that if we consider $-\vec{v}$, the rotation would be still be 2π (why?).

Consider $\vec{v} = (y, -x)$. Now $x' = y, y' = -x$ and hence $(x, y) = (x_0, y_0)e^{-it}$. Thus every orbit away from $(0, 0)$ is periodic. Indeed, the vector field "curls" around the origin. It has rotation 2π . (By the way, this "curl" business seems to suggest that Green's theorem might play a role and indeed it does.)

2. Consider a unit circle around $(-2, 0)$ and $\vec{v} = (x, y)$. What is the rotation? (It is zero!)

It seems that a periodic orbit always encloses an equilibrium. Indeed, we shall see that this is true. We now define the Poincaré index of a smooth vector field with isolated equilibrium points with respect to a piecewise smooth piecewise regular Jordan curve: Suppose γ does not pass through any equilibrium point. The index is $I_{\vec{v}}(\gamma) = \frac{1}{2\pi} \int_a^b \frac{v_1 v_2' - v_2 v_1'}{v_1^2 + v_2^2} ds$ where γ is parametrised in the "anticlockwise direction", i.e., $\hat{k} \times \gamma'$ points inside.

This definition is reasonable because we can prove the existence of a continuous piecewise smooth function $\theta : [a, b] \rightarrow \mathbb{R}$ such that $\vec{v}(\gamma(t)) = \|\vec{v}\|(\gamma(t))(\cos(\theta), \sin(\theta))$. Indeed, the assumption of no equilibria along the curve implies this fact by using the notion of a continuous lift (just as in the Prüfer substitution). Now $\theta' = \frac{v_1 v_2' - v_2 v_1'}{v_1^2 + v_2^2}$ and hence the index is precisely the “number of times” the vector field “winds” around. We can now calculate the index for the above examples (as well as say for (x^2, y^2) around the unit circle) and verify that our expectations are correct.