# 1 Recap

1. Poincaré index $I_v(\gamma) = \frac{1}{2\pi}\int_a^b \frac{v_1 v_2' - v_2 v_1'}{v_1^2+v_2^2} ds$ and examples/

# 2 Periodic orbits

Theorem: Let $\gamma$ be a piecewise regular piecewise smooth Jordan curve such that it and its interior do not contain any equilibria. Then $I_{\vec{v}}(\gamma) = 0$.
Proof: By Green's theorem, we see the result holds. □
In fact, the above result holds for any bounded domain whose boundary is a finite union of piecewise regular piecewise smooth Jordan curves provided they are parametrised in the "right" direction. Using this observation, we can talk of the index of an isolated equilibrium without reference to any specific Jordan curve (because we can prove that the answer does not depend on the curve).
We can prove a more general result: If $\gamma$ is a piecewise smooth piecewise regular Jordan path whose interior contains only finitely many equilibria of $\vec{v}$ (that does not vanish on the image of $\gamma$), then the index of $\vec{v}$ is the sum of indices of each equilibrium.
Now note that the index can be defined even if the vector field is defined on the image of the Jordan curve. Using this observation, we can ask what the index of a curve with respect to its own tangent vector is, i.e., how much does the tangent vector rotate?

**Theorem 1.** *Let $\gamma$ be a $C^2$ Jordan curve which is regular and $\gamma'(a) = \gamma'(b)$. Then $I_{\gamma'}(\gamma) = 1$.*

*Proof.* Firstly, we can assume that $\gamma : [0, l] \to \mathbb{R}^2\ 0$ has unit speed by reparametrisation ((does not change the index - why?)). Secondly, by rigid motions and reparametrisation, we can assume that $\gamma$ is in the upper half-plane, and $\gamma(0)_2 = \gamma_2(l) = 0$ (with $\gamma_2 \geq 0$ otherwise).

**Remark**: In fact, we can ensure that $\gamma_2 > 0$ away from a single point. Indeed, we consider the convex hull (the intersection of all convex sets containing our set) $K$ of $\gamma[0, l]$. This set is compact and convex. (Indeed, take all convex linear combinations of finitely many elements of $\gamma[0, l]$. This is the convex hull.) It has a non-empty interior. Indeed, a Jordan curve cannot lie on a line (why?) and hence $K$ has non-empty interior (why?) Take an interior point $x_0$. By means of looking at the "last" point of intersection of a ray from $x_0$, we can prove that the boundary $C$ of $K$ is a Jordan curve. Now taking a countable dense subset of $S^1$, we can show that along most directions $u$, the linear functional $\langle x, u\rangle$ achieves a minimum on $C$ at a single point $p$. Now $p = \sum_i \lambda_i x_i$ where $x_i \in \gamma[0, l]$. Note that $\langle x_i, u\rangle \geq p$ and hence $x_i = p\ \forall\ i$. Thus $p \in \gamma[0, l]$. Now rotate and translate so that $p$ is at the origin. Reparametrise so that $\gamma(0) = \gamma(l) = p$.

The idea is to compare the rotation of the tangent vector with that of the secant/chord around the $x$-axis. So consider the secant vector field $X(s, t)$ defined on the triangular region $0 \leq s \leq t \leq l$ as $X(s, s) = \gamma'(s)$, $X(s, t) = \frac{\gamma(t)-\gamma(s)}{\|\gamma(t)-\gamma(s)\|}$ when $t > s$, $(s, t) \neq (0, l)$ and $X(0, l) = -\gamma'(0)$. This vector field is continuous (why?). Let $\theta(s, t)$ be the angle made by $X(s, t)$ with the positive $x$-axis. (It is well-defined and continuous by the lifting property.) Choose the initial point such that $\theta(0, 0) = 0$.

Clearly, $\theta(0,t)$ varies from $0$ to $\pi$ as $t$ varies from $0$ to $l$. On the other hand $\theta(s,l)$ varies from $\pi$ to $2\pi$. Thus $\theta(s,s)$ changes by $2\pi$ on $[0,l]$. This means the index is $1$. $\square$