

# 1 Recap

1. No equilibria implies zero Poincaré index.
2. Definition of  $I_{\gamma'}(\gamma)$  and stated that it is 1 for regular  $C^2$  Jordan curves with  $\gamma'(a) = \gamma'(b)$ .
3. Proved reparametrisation and rigid-motion invariance of  $I_{\gamma'}(\gamma)$ .

# 2 Periodic orbits

*Proof.* Firstly, we can assume that  $\gamma : [0, l] \rightarrow \mathbb{R}^2$  has unit speed by reparametrisation. Secondly, by rigid motions and reparametrisation, we can assume that  $\gamma$  is in the upper half-plane, and  $\gamma(0)_2 = \gamma_2(l) = 0$  (with  $\gamma_2 \geq 0$  otherwise) - why?.

**Remark:** In fact, we can ensure that  $\gamma_2 > 0$  away from a single point. Indeed, we consider the convex hull (the intersection of all convex sets containing our set)  $K$  of  $\gamma[0, l]$ . This set is compact and convex. (Indeed, take all convex linear combinations of finitely many elements of  $\gamma[0, l]$ . This is the convex hull.) It has a non-empty interior. Indeed, a Jordan curve cannot lie on a line (why?) and hence  $K$  has non-empty interior (why?) Take an interior point  $x_0$ . By means of looking at the "last" point of intersection of a ray from  $x_0$ , we can prove that the boundary  $C$  of  $K$  is a Jordan curve. Now taking a countable dense subset of  $S^1$ , we can show that along most directions  $u$ , the linear functional  $\langle x, u \rangle$  achieves a minimum on  $C$  at a single point  $p$ . Now  $p = \sum_i \lambda_i x_i$  where  $x_i \in \gamma[0, l]$ . Note that  $\langle x_i, u \rangle \geq p$  and hence  $x_i = p \forall i$ . Thus  $p \in \gamma[0, l]$ . Now rotate and translate so that  $p$  is at the origin. Reparametrise so that  $\gamma(0) = \gamma(l) = p$ .

The idea is to compare the rotation of the tangent vector with that of the secant/chord around the  $x$ -axis. So consider the secant vector field  $X(s, t)$  defined on the triangular region  $0 \leq s \leq t \leq l$  as  $X(s, s) = \gamma'(s)$ ,  $X(s, t) = \frac{\gamma(t) - \gamma(s)}{\|\gamma(t) - \gamma(s)\|}$  when  $t > s$ ,  $(s, t) \neq (0, l)$  and  $X(0, l) = -\gamma'(0)$ . This vector field is continuous (why?). Let  $\theta(s, t)$  be the angle made by  $X(s, t)$  with the positive  $x$ -axis. (It is well-defined and continuous by the lifting property.) Choose the initial point such that  $\theta(0, 0) = 0$ . Clearly,  $\theta(0, t)$  varies from 0 to  $\pi$  as  $t$  varies from 0 to  $l$ . On the other hand  $\theta(s, l)$  varies from  $\pi$  to  $2\pi$ . Thus  $\theta(s, s)$  changes by  $2\pi$  on  $[0, l]$ . This means the index is 1.  $\square$

As a consequence, we see that if  $C$  is a periodic orbit, then the index is 1 and if the equilibria are isolated, then  $C$  contains only finitely many equilibria whose indices sum up to 1 (and hence in particular, there is at least one equilibrium inside  $C$ ). But this criterion is only specific to two dimensions.

Example: Consider  $x' = y, y' = -x, z' = 1 - x^2 - y^2$ . This has no equilibria and yet  $(\cos(t), \sin(t), c)$  are periodic orbits