1 Recap

1. If there is a periodic orbit, it encloses at least one equilibrium point (in 2D if the points are isolated).

Now we want criteria for the existence or the lack thereof of periodic orbits. (Obviously $v = (f, g)$ must have equilibria. This is already an easy condition to verify.)
Bendixon's criterion: If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ has a definite sign in Ω which is simply connected (assume (f, g) is C^1), then Ω has no periodic orbit.
Proof: If Ω has a periodic orbit C, note that on $[0, T]$, C is a Jordan curve that is regular. Let D be interior to C. Apply Green to see that $\int_C (f dy - g dx)$ has a sign but also vanishes - a contradiction.
Here is the Poincaré-Bendixon theorem:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain together with its boundary. Suppose it does not contain any equilibrium (typically we think of Ω as an " annular" region). If C is an orbit in Ω for $t \geq t_0$, then either C is periodic or it "tends" to a periodic one as $t \to \infty$ (that is, the distance between $\gamma(t)$ and the periodic one goes to 0).*

In other words, either orbits are unbounded or bounded. If you take a bounded orbit, either it converges to equilibrium or to a periodic orbit. Here is an example:
Let $x' = -y + x(1 - x^2 - y^2), y' = x + y(1 - x^2 - y^2)$. We want to consider periodic orbits. To this end, firstly, the only equilibrium is $(0, 0)$. Consider an annulus $\Omega = \{\frac{1}{2} \leq r \leq \frac{3}{2}\}$. It does not contain equilibria but "surrounds" one. If we find some orbit positively contained in this annulus, then there is a periodic orbit. To this end, let $r^2 = x^2 + y^2$. Then $r' = r(1 - r^2)$. We can easily see that $r = 1$ is a periodic orbit. We can also deduce the existence of a periodic orbit using Poincare'e-Bendixon. (By the way, we can easily see using Bendixon's criterion that there are no periodic orbits very close to the origin.) In fact, we can solve explicitly to conclude that $r = (1 + c\exp(-2t))^{-1/2}$ where $c = \frac{1-r_0^2}{r_0^2}$. Thus every orbit spirals towards the only periodic one.
For the particular case of second order equations, we have a better result due to Leinard: Consider $x'' + f(x)x' + g(x) = 0$ where f, g satisfy

1. f, g are C^1 functions on $\mathbb{R}$,
2. g is odd, $g > 0$ for $x > 0$, and f is even.
3. The odd function $F(x) = \int_0^x f(s)ds$ has exactly one positive zero a, $F < 0$ on $(0, a)$ and > 0 on (a, ∞), F is increasing in (a, ∞), and $F(x) \to \infty$ as $x \to \infty$.

Then there is a unique periodic orbit surrounding the only equilibrium point $(0, 0)$. Moreover, every other orbit approaches this one as $t \to \infty$.
Example: $x'' + \mu(x^2 - 1)x' + x = 0$ with $\mu > 0$ produces a periodic orbit.

2 Stable manifold theorem - Proof

Here is a high-level idea: Basically, $x' \approx Ax$ where $A = Df_0$. Now collecting the generalised eigenspaces with negative and positive real parts (respectively), we see that $B = CAC^{-1}$ is block diagonal (with *Real* blocks P, Q by the *Real* Jordan canonical form theorem) and $y = Cx$ satisfies $y' \approx By$. Now roughly speaking, we want to solve for $y_{k+1}, \ldots, y_n$ in terms of $y_1, \ldots, y_k$ (such that the solution is close to the plane $y_{k+1} = 0 = \ldots$). The idea is to go through the usual proof of existence by successive approximations carefully (using the Duhamel formula treating the error term as the forcing term), and start with initial data such that we expect the solution to go to 0 as $t \to \infty$. Then it turns out that such an initial data set is naturally a graph locally near the origin over $y_1, \ldots, y_k$. We then prove that if we do not start on this set, we cannot go to infinity. Using this we prove that S is invariant. For U, we simply change the direction of time and apply the same argument.

To go a little further into detail, note that $y' = By + G(y)$ where because F is C^1, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x\| \leq \delta, \|y\| \leq \delta, \|G(y) - G(x)\| \leq \epsilon\|x - y\|$. We shall choose ϵ later. Let $U(t)$ be block diagonal with the upper block being e^{Pt} and everything else being zero. Likewise $V(t)$ has e^{Qt} on the lower block. Note that $e^{Bt} = U + V$. The Duhamel principle states that

$$y(t) = U(t)y(0) + \int_0^t U(t-s)G(y(s))ds + V(t)y(0) + \int_0^t V(t-s)G(y(s))$$

which (for ease of grouping the stable and unstable terms) is

$$y(t) = U(t)y(0) + \int_0^t U(t-s)G(y(s))ds + V(t)(y(0) + \int_0^\infty V(-s)G(y(s))ds) - \int_t^\infty V(t-s)G(y(s))ds.$$

Note that the integral from 0 to ∞ is expected to exist (why?) All the terms except for $V(t)(y(0) + \int_0^\infty V(-s)G(y(s))ds)$ are expected to go to 0 as $t \to \infty$. Thus for stability to hold, we better require $(y(0) + \int_0^\infty V(-s)G(y(s))ds)$ to be constant and have the last few components as 0. Indeed, we turn this around and state our mission as solving the integral equation

$$y(t,a) = U(t)a + \int_0^t U(t-s)G(y(s,a))ds - \int_t^\infty V(t-s)G(y(s,a))ds$$

with $a_{k+1} = a_{k+2} = \ldots = 0$. (We want to choose $y_i(0) = \psi_i(a_1, \ldots, a_k) = -\int_0^\infty V(-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds$ when $i \geq k+1$.) In other words, the initial values of y_i (for $i \geq k+1$) are hopefully graphs of functions of $y_1(0), \ldots$. Thus if we manage to solve the equation on $\|(y_1, \ldots, y_k)(0)\| < \delta'$ for some δ', then on this set, we have a manifold of dimension k given as a graph. We shall prove that this manifold is invariant and that it is indeed the stable manifold.

To solve the integral equation for small values of $(y_1(0), \ldots, y_k(0))$, we use an iteration with $y^{(0)} = 0$. Then define

$$y^{(j+1)}(t,a) = U(t)a + \int_0^t U(t-s)G(y^{(j)}(s,a))ds - \int_t^\infty V(t-s)G(y^{(j)}(s,a))ds.$$

As in the usual proof of existence (Picard's theorem), we want to prove that successive iterates get close to each other. Note that

$$\|y^{(j+1)}(t,a) - y^{(j)}(t,a)\| \leq \int_0^t \|U(t-s)\|\|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\|ds + \int_t^\infty \|V(t-s)\|\|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\|ds. \quad (1)$$

At this point note that $\|U(t)\| \leq Ke^{-(\alpha+\sigma)t}\ \forall\ t \geq 0$ and $\|V(t)\| \leq Ke^{\sigma t}\ \forall\ t \leq 0$ for some constants $K, \alpha, \sigma > 0$. (By the HW exercise.) Thus

$$\|y^{(j+1)}(t,a) - y^{(j)}(t,a)\| \leq K \int_0^t e^{-(\alpha+\sigma)(t-s)} \|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\| ds$$
$$+ \int_t^\infty Ke^{\sigma(t-s)} \|G(y^{(j)}(s,a)) - G(y^{(j-1)}(s,a))\| ds. \tag{2}$$

This is one more reason to have the limits from t to ∞ in the second integral. We assume inductively that $\|y^{(j)} - y^{(j-1)}\| \leq C_j e^{-\mu t}$ for appropriately chosen C_j and μ. Upon substitution, we see that the induction step can be completed (for instance) if we assume that $C_j = \frac{C}{2^{j-1}}$ (for some C to be chosen later) and $\mu = \alpha$ if $\epsilon < \frac{\sigma}{4K}$ and $\|y^{(j)}\| < \delta$ for all j. Since the induction hypothesis must be met for $j = 1$, we see that $C = K\|a\|$ works. The condition involving δ can be met (inductively) if $\|a\| < \frac{\delta}{2K}$. We see that we get a Cauchy sequence that converges uniformly in t (for all $t \geq 0$) to a continuous function y. By uniform convergence (how), we see that the integral equation is met for all t and hence by the fundamental theorem of calculus, y is C^1. It also satisfies $\|y\| \leq 2K\|a\|e^{-\sigma t}$ for $\|a\| < \delta' = \frac{\delta}{2K}$.

From the integral equation and the initial data, we see that the ODE is satisfied. Moreover, $y_i(t) = -\int_t^\infty V(t-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds$ for $i \geq k+1$. The graph S given by $(a_1, \ldots, a_k, \psi(a_1, \ldots, a_k) = -\int_0^\infty V(-s)G(y(s, a_1, \ldots, a_k, 0, \ldots))ds)$ turns out to be C^1. Indeed, it is not hard to prove that y depends on $a_1, \ldots$ in a C^0 manner by using the successive approximations and induction. To prove that it is C^1 requires more work (and the idea is similar to the proof of differentiable dependence on parameters). One can also show that S is tangent to the stable subspace at the origin.

For $\|a\| < \delta'$, we show that a solution of the integral equation above satisfying $\|y\| < \delta'$ is unique and hence the one we constructed is *the* solution: Suppose $\tilde{y}$ is another such solution. Then $\|\tilde{y} - y\| \leq 2\frac{K}{\sigma}\epsilon M$ where M is the supremum of $\|\tilde{y} - y\|$. Thus we have a contradiction unless $M = 0$.

We see using the estimates that as $t \to \infty$, $y \to 0$ if the initial data lies on S. We need to now prove that if we start with a point close to the origin and not on S, then $\|y\|$ cannot be $\leq \delta$ for all $t \geq 0$. This also proves that S is invariant (why?). Indeed we now prove that if $y(0) \notin S$ and is small, then $\|y(t)\| \leq \delta$ cannot be met for all $t \geq 0$: Simply use the Duhamel formula to get a contradiction as $t \to \infty$.