1 Recap

1. Defined the Wronskian and proved that its non-vanishing can be used to prove linear independence.
2. Defined the matrix exponential and proved some properties (including the ability to calculate it using Jordan blocks).

2 Linear systems of ODE

Another interesting application of the Wronskian: If $y'' = a(x)y' + b(x)y$, then $W(x)$ the Wronskian, satisfies $W' = aW$ and hence W is known. (As a consequence, if the Wronskian vanishes at one point, it does so everywhere.) Now we can solve for one of the linearly independent solutions knowing the other!

We have one last property of matrix exponentials:

1. $\det(e^A) = e^{tr(A)}$: $\det(e^A) = \det(Pe^J P^{-1}) = \det(e^J) = e^{\sum_i \lambda_i} \det(e^{N_1}) \det(e^{N_2}) \ldots$. Now e^N is upper-triangular with diagonal entries equal to 0 (why?) and thus $\det(e^N) = 1$. Thus we are done.

Now we define a matrix-valued function of 1-variable to be continuous/differentiable iff each entry is so. We claim that if $A(t), B(t)$ are differentiable then so is AB and $(AB)' = A'B + AB'$ (exercise). Moreover,
Lemma: e^{tA} is differentiable and its derivative is Ae^{tA}.
Proof: $e^{tA} = Pe^{tJ}P^{-1}$. Now for a pure Jordan block J_i, $e^{tJ_i} = e^{t\lambda}e^{tN_i}$. Now e^{tN_i} has polynomial entries in t and is hence differentiable. Thus e^{tA} is differentiable. Now $e^{tN_i} = I + tN_i + t^2N_i^2/2! + \ldots + t^{k-1}N_i^{k-1}/(k-1)!$ and hence $(e^{tN_i})' = N_i(1 + tN_i + \ldots) = N_i e^{tN_i}$. Hence $(e^{tJ_i})' = J_i e^{tJ_i}$ and $(e^{tA})' = Ae^{tA}$.
Now we can solve $y' = Ay$: Note that $(e^{-At}y)' = e^{-At}y' - e^{-At}Ay = 0$ and hence $e^{-At}y = y_0$ and $y = e^{At}y_0$. Here is a basis of solutions: $e^{At}e_i$. The space of solutions is n-dimensional. We can calculate the matrix exponential using the Jordan canonical form (or any other method of our choice).
We can also solve the inhomogeneous autonomous problem $y' = Ay + f$: $e^{-At}(y' - Ay) = e^{-At}f$ and hence $(ye^{-At})' = e^{-At}f$ and $y = e^{At}y_0 + e^{At}\int_0^t e^{-As}f(s)ds$ or more generally, $y = e^{A(t-t_0)}y(t_0) + e^{A(t-t_0)}\int_{t_0}^t e^{-As}f(s)ds$. This formula is called the Duhamel formula/method of variation of parameters.
Here is an example: $y'' = -ky - by' + f$. We already know what e^{At} is in this case (why?) Now we have reduced the problem to an integral involving f. If f is exponential, the integral is easy. Otherwise, one cannot do much more.
In older parlance, if we know one solution (a "particular" solution) to $y' = Ay + f$, then every solution is $y = y_0 + h$ where h is a solution of the homogeneous problem (given by $e^{At}y_0$). Since the particular solution is obtained by simply making y_0 into a specific function (as opposed to a constant), the name "variation of parameters" came about.