1 Recap

1. Determinant of exponentials.

2. Solution of homogeneous and inhomogeneous autonomous problems (Duhamel).

2 Linear systems

We shall now discuss only uniqueness (existence will be considered later) for a non-autonomous system $y' = A(t)y + B(t)$ where $A(t), B(t)$ are continuous functions on $[a, b]$ with $y(t_0) = y_0$ where $t_0 \in [a, b]$. Without loss of generality, $t_0 = 0$ (why?). Suppose u, v are differentiable solutions with $u(t_0) = v(t_0)$, then $(u - v)' = A(t)(u - v)$. Thus $u - v = \displaystyle\int_{t_0}^{t} A(u - v)ds$. Thus $\|u - v\| = \| \displaystyle\int_{t_0}^{t} A(u - v)ds$. Now we have the following lemma.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}^n$ be a continuous function. Then $\| \int_{t_0}^{t} f(s)ds\| \leq \int_{t_0}^{t} \|f(s)\|ds$.*

Proof. Indeed, let $v(t) = \int_{t_0}^{t} f(s)ds$. Then $\|v\|^2 = v. \int_{t_0}^{t} f(s)ds = \int_{t_0}^{t} v(t).f(s)ds \leq \int_{t_0}^{t} \|v(t)\|\|f(s)\|ds = \|v(t)\| \int_{t_0}^{t} \|f(s)\|ds$. Hence we are done. □

Thus, $\|u - v\| \leq \int_{t_0}^{t} \|A(t)\|\|u - v\|$. If $A(t)$ is continuous, then so is $\|A(t)\|$ and thus $\|A(t)\| \leq C$ on $[a, b]$. Using Gronwall's inequality (HW problem), we see that $\|u - v\| \leq e^{C(t-t_0)}\|u - v\|(t_0) = 0$. Hence $u = v$. □

For the inhomogeneous non-autonomous case, $y' = A(t)y + B(t)$, Suppose we solve the homogenous system $y' = A(t)y$ with $y(t_0) = e_i(t_0)$ (note that we already know uniqueness, and we are assuming existence), we get a bunch of solutions that are linearly independent (why?) and if arrange them in a column, we get an invertible matrix $\Phi(t, t_0)$. Let $\Psi(t)$ be any invertible matrix satisfying $\Psi' = A(t)\Psi$. Note that $\Phi(t, t_0) = \Psi(t)\Psi(t_0)^{-1}$ (why?) As a consequence, $\vec{y}(t) = \Phi(t, t_0)\vec{y}(t_0)$ and hence the space of solutions is n-dimensional.
We claim that the solution to $y' = A(t)y + B(t)$ is $y(t) = \Phi(t, t_0)y_0 + \int_{t_0}^{t} \Phi(t, s)B(s)ds$ akin to the Duhamel formula for the autonomous case.

Proof. Let's differentiate this formula and check that it satisfies the equation (it obviously satisfies the initial conditions).

$$\begin{aligned} y' &= A\Phi(t, t_0)y_0 + \frac{d}{dt}\int_{t_0}^{t} \Phi(t, s)B(s)ds = Ay + \Psi' \int_{t_0}^{t} \Psi(s)^{-1}B(s)ds \\ &= A\Phi(t, t_0)y_0 + A\Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)B(s)ds = Ay. \end{aligned} \tag{1}$$

□

3 Real-analytic functions

The method of exponentiation tells us that it might be prudent to try to solve ODE using power series. To this end, we make a definition:
Def: A function $f : (a, b) \to \mathbb{R}$ is said to be real-analytic at t_0 if there exists a $\delta > 0$ such that $f(t) = \sum_{k=0}^{\infty} c_k (t - t_0)^k$ converges for all $t \in (t_0 - \delta, t_0 + \delta)$. It is said to be real-analytic on (a, b) if it is real-analytic at every point in (a, b).
Before we come up with examples, here are some important facts from analysis:

1. The Ratio test: Let $L = \limsup \frac{|a_{n+1}|}{|a_n|}$ and l be the $\liminf$. If $L < 1$ then $\sum a_n$ converges absolutely. If $l > 1$, it diverges.

2. The Root test: Let $L = \limsup |a_n|^{1/n}$. If $L < 1$ then $\sum a_n$ converges absolutely. If $L > 1$ it diverges.

3. Applying the root test to power series we see that if $R^{-1} = \limsup |a_n|^{1/n} > 0$, then $\sum a_n x^n$ converges absolutely when $|x| < R$ (and uniformly on any compact subset of the disc of convergence) and diverges when $|x| > R$. This R is called the radius of convergence. (The ratio test can also be used to determine R in many cases.)

4. On a compact subset of the disc of convergence, the power series can be differentiated and integrated term-by-term to get a power series that also converges uniformly.