1 Recap

1. Duhamel formula for non-autonomous equations.

2. Review of power series.

2 Real-analytic functions

Examples/counterexamples:

1. $e^x = 1 + x + x^2/2! + \ldots$ has infinite radius of convergence. It is certainly real-analytic at $x = 0$.

2. $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ has radius of convergence 1. It is real-analytic on the disc of convergence. In fact, $\frac{1}{1-a-(x-a)} = \frac{1}{1-a}(1 + \frac{x-a}{1-a} + \ldots)$ has radius of convergence $|1-a|$. It is real-analytic everywhere except at 1. Such a point is called a singular point.

3. Let $f(x) = e^{-1/x^2}$ when $x > 0$ and $f(x) = 0$ when $x \leq 0$. We claim that this function is smooth, i.e., it is differentiable everywhere as many times as we want. But it is not real-analytic! (why?)
   Proof of Claim: f is obviously smooth everywhere except possibly at 0. It is continuous at 0 (why?) $f' = \frac{2}{x^3}f$ when $x \neq 0$. We can easily see that f is differentiable at 0 and in fact f' is continuous at 0 too. This gives us a hint by induction. Assume the induction hypothesis that $f^{(k)} = fp_k(\frac{1}{x})$ when $x \neq 0$ and 0 when $x = 0$ where p_k is a polynomial of degree at most $3k$. Now when $x \neq 0$, $f^{(k+1)} = \frac{2}{x^3}fp_k(1/x) - \frac{1}{x^2}fp_k'(1/x)$. Clearly $f^{(k+1)}$ is also of the same form. Moreover, it is 0 at 0 (why?).

We now prove a useful lemma.

Lemma 2.1. *If $f : I = (t_0 - \delta, t_0 + \delta)$ is a uniformly convergent power series around t_0, then f is real-analytic at every other point, i.e., it can be locally expressed as a power series around any other point in the interval.*

Proof. Consider $a \in I$. Then $f(x) = \sum (x - a - (t_0 - a))^n c_n$ where this series converges absolutely and uniformly on $|x - a| < \delta - |t_0 - a|$ by the triangle inequality. Expanding, $f(x) = \sum_n \sum_{k=0}^n \binom{n}{k}(x-a)^k(t_0-a)^{n-k}c_n$. By absolute convergence, we can sum in any order we want and hence we interchange the summation to get $f(x) = \sum_{k=0}^\infty (x - a)^k \sum_{n=k}^\infty c_n(t_0-a)^{n-k}$. □

Using this lemma we prove the following interesting characterisation of real-analytic functions.

Theorem 1. *A real-valued function f defined in a neighbourhood of t_0 is real-analytic at t_0 iff*

1. *f is smooth in a neighbourhood of t_0, and*

2. *there exist positive δ, M such that for $t \in (t_0 - \delta, t_0 + \delta)$, $|f^{(k)}(t)| \leq M\frac{k!}{\delta^k}$ for $k = 0, 1, 2 \ldots$.*

Proof.

1. If the conditions are met: By Taylor's theorem, the error term in the Taylor series up to order $k-1$ (here we use the smoothness of f in a neighbourhood of t_0) is $f^{(k)}(t_k)\frac{(x-t_0)^k}{k!}$ where $t_k \in [t_0-\delta/2, t_0+\delta/2]$. This error is at most $\frac{M}{2^k}$. Hence as $k \to \infty$ we see that the Taylor series converges uniformly on $[t_0-\delta/2, t_0+\delta/2]$. Hence f is real-analytic at t_0.

2. If f is real-analytic at t_0: HW (Hint: Use the proof of the lemma above).

□

If we replace $k!$ by $(k!)^s$ where $s \geq 1$, we get a class of smooth functions called the Gevrey class of order s. This class (and its generalisation to multivariables) is very important for PDE of various types. Cedric Villani won the fields medal for using this class effectively for a specific PDE (the Boltzmann equation). Here are a couple of examples of using power series for ODE (before we state and prove general results).

1. $y'' + y = 0$: Suppose there exists a twice-differentiable function satisfying this equation. Then by induction we see that it is smooth and $|y^(k+2)| = |y^(k)| = \ldots$ and hence the conditions of the theorem above are met (why is the second one met?). Thus y is real-analytic. Now $y = \sum_n c_n t^n$ and hence $y'' = \sum_n n(n-1)c_n t^{n-2}$. Thus $c_n + c_{n+2}(n+1)(n+2) = 0$ for $n \geq 0$. Given c_0, c_1, we can determine all the other coefficients inductively. The power series turns out to be $y = y_0 \sum (-1)^n \frac{t^{2n}}{(2n)!} + c_1 \sum (-1)^n \frac{t^{2n+1}}{(2n+1)!)}$ which is expected.

2. $y'' - 2yt' + 2py = 0$ where $p \in \mathbb{R}$ (the Hermite equation): Again, if y is twice differentiable, then from the equation it is smooth (why?). We know that if there is a solution it is unique given y, y' at $t = 0$. Now we try to find it using power series. $y = \sum a_n t^n$. It satisfies $(n+1)(n+2)a_{n+2} = -2(p-n)a_n$. Thus $y = a_0 y_1 + a_1 y_2$. It is easy to check that y_1, y_2 are convergent on $\mathbb{R}$, and that they are linearly independent. If $p = 0, 1, 2, 3, \ldots$ one of these solutions becomes a polynomial. (These polynomials are called Hermite polynomials and satisfy $H_n = (-1)^n e^{t^2} \frac{d^n(e^{-t^2})}{dt^n}$.) These polynomials also occur in quantum mechanics as the eigenstates (after multiplying by a Gaussian) of the harmonic oscillator. They are also used in numerical integration (Gauss-Hermite quadrature).