1 Recap

1. Examples and counterexamples for real-analyticity.
2. Characterisation of real-analytic functions
3. Solving the (approximation of the) simple pendulum using power series.

2 Real-analytic functions

We now prove a general result.

Theorem 1. *Consider* $\vec{y}' = A(t)\vec{y}(t) + B(t)$ *where* A, B *are real analytic matrix-valued functions at* $t = t_0$ *such that they are power series on* $(t_0 - R, t_0 + R)$*. Then there exists a unique solution to this equation with* $\vec{y}(t_0), \vec{y}'(t_0)$ *specified. Moreover, the solution is real-analytic at* $t = t_0$ *with radius of convergence at least* R*.*

Proof. Uniqueness: We have already proven uniqueness of differentiable solutions given $y(t_0)$ using the Gronwall inequality approach. By the way, any differentiable solution is smooth on $(t_0 - R, t_0 + R)$ (why?).
Existence: Without loss of generality, assume that $t_0 = 0$. The radius of convergence of A and B is at least R. Let $r < R$ be fixed but arbitrary. The power series $A(t) = \sum A_n t^n$ and $B(t) = \sum \vec{b}_n t^n$ converge uniformly on $[-r - \epsilon, r + \epsilon]$. We shall try to produce a real-analytic solution that uniformly converges on $[-r, r]$ as $\vec{y}(t) = \sum \vec{c}_n t^n$. Then

$$(n+1)\vec{c}_{n+1} = \sum_{k=0}^{n} A_k \vec{c}_{n-k} + \vec{b}_n \; \forall \, n \geq 0. \tag{1}$$

Inductively, we can solve for $\vec{c}_n$ uniquely given $\vec{c}_0$. The question is whether we get a convergent power series. We see that

$$(n+1)\|\vec{c}_{n+1}\| \leq \sum_{k=0}^{n} \|A_k\|\|\vec{c}_{n-k}\| + \|\vec{b}_n\|. \tag{2}$$

Since A, B are uniformly convergent power series, $\|A_k\| \leq \frac{M}{(r+\epsilon)^k}$ and $\|\vec{b}_n\| \leq \frac{M}{(r+\epsilon)^k}$. Let $\tilde{r} = r + \epsilon$. Let u_n satisfy the following identities.

$$(n+1)u_{n+1} = \sum_{k=0}^{n} \frac{M}{\tilde{r}^k} u_{n-k} + \frac{M}{\tilde{r}^n}. \tag{3}$$

If $u_0 = \|\vec{c}_0\|$, then $u_n \geq \|\vec{c}_n\| \; \forall \, n \geq 0$ (why?). Writing the same as above for n,

$$n u_n = \sum_{k=0}^{n-1} \frac{M}{\tilde{r}^k} u_{n-1-k} + \frac{M}{\tilde{r}^{n-1}}. \tag{4}$$

Thus

$$
\begin{aligned}
(n+1)u_{n+1} &= Mu_n + \frac{nu_n}{\tilde{r}} \\
\Rightarrow \frac{u_{n+1}}{u_n} &= \frac{M\tilde{r}+n}{(n+1)\tilde{r}}
\end{aligned}
\tag{5}
$$

The limit is $\frac{1}{\tilde{r}}$. Thus we are done (why?). □

While the theorem above is pleasant, it does not cover all cases of interest. For example, suppose we want to study an electrostatic field in a long cylinder with potential (cylindrical symmetry) specified on the cylinder, we will have to solve the Laplace equation in cylindrical coordinates (r, θ, z). That is,

$$
\frac{1}{r}\partial_r(r\phi_r) + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0. \tag{6}
$$

We use the method of separation of variables, i.e., $\phi = R(r)P(\theta)Z(z)$. Then we see that R satisfies

$$
\frac{r}{R}R' + \frac{r^2}{R}R'' - \lambda r^2 = constant. \tag{7}
$$

That is, it is of the form (after changing variables and solving the other equations) $y'' + \frac{y'}{t} + y\left(1 - \frac{\nu^2 s}{t^2}\right) = 0$ where ν is an integer. This is an example of Bessel's equation. (The solutions are called Bessel functions $J_\nu(t)$.) It occurs in various other situations in real life (and in probability I believe as the pdf of a product of two normal variables). Likewise, if we try to solve the Laplace in spherical coordinates (or more generally, the eigenvalues problem arising from the Hydrogen atom for instance), after separation of variables, the solutions to the angular part are $P_n(\cos(\theta))$ where P_n are the Legendre polynomials satisfying the equation

$$
(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{8}
$$

Suppose we declare $v = y'$, these equations do not fall under the purview of the theorem above because A, B are no longer real-analytic. They have singularities (why?) In fact, if we try $v = ty'$, then the equations fall under the purview of

$$
\vec{y}' = \frac{A(t)}{t}\vec{y}, \tag{9}
$$

where $A(t)$ is real-analytic. Such systems are called systems with *regular singular points* (an oxymoron?)