1 Recap

1. Stated and proved a theorem about real-analytic non-autonomous linear systems having real-analytic solutions.

2. Motivated Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ through the Hydrogen atom and started $\vec{y}' = \frac{A(t)}{t}\vec{y}$ where $A(t)$ is real-analytic. Such systems are called systems with *regular singular points*

2 Real-analytic functions

Suppose $\vec{y}(t_0) = \vec{y}_0$ where $t_0 > 0$ (the case of < 0 is similar), then declare $s = \ln(t)$. We can now consider

$$\frac{d}{ds}\vec{y}(e^s) = A(e^s)\vec{y}(e^s). \tag{1}$$

The usual Frobenius method above can be applied to this equation to conclude the existence of a real-analytic solution in a neighbourhood of $s_0 = \ln(t_0)$. Changing variables, we see that y is a power series in $\ln(t)$. Now here is an interesting lemma (proof is a HW exercise).

Lemma 2.1. *Let $f(t)$ be real-analytic at $g(t_0)$ and $g(t)$ at t_0. Then $h(t) = f(g(t))$ is real-analytic at t_0.*

Thus y is real-analytic in t near t_0. However, what can we say about the power series really? For instance, if $A(t)$ is a constant, then in terms of s, $\vec{y} = e^{A(s-s_0)}\vec{y}_0$. Thus, $\vec{y}(t) = e^{A\ln(t)}e^{-A\ln(t_0)}\vec{y}_0$. Now writing A in the Jordan canonical form (and noting that the ratio test and so on apply to complex power series as well), $e^{A\ln t} = Pe^{J\ln(t)}P^{-1}$. If J is a Jordan block, then $e^{J\ln(t)} = t^{\lambda_R}(\cos(\lambda_I\ln(t)) + \sqrt{-1}\sin(\lambda_I\ln(t)))(I + N\ln(t) + N^2(\ln(t))^2/2! + \ldots)$. Thus we can in general hope for a solution involving linear combinations of functions of the type $t^r\sum a_n t^n(p(\ln(t))$ where r and a_n are in general, complex, and p is a possibly complex polynomial.
Let us specialise to equations of the form $y'' + P(t)y' + Q(t)y = 0$. If we want this to have regular singular points at $t = 0$, then $P(t)t$ and $Q(t)t^2$ are real-analytic around 0 (why?) Thus $P(t) = \frac{p_0}{t} + p_1 + p_2 t + \ldots$, $Q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + \ldots$. Let $y(t) = t^m(a_0 + a_1 t + \ldots)$. Then $y' = \sum_{n=0} a_n(m+n)t^{m+n-1}$, $y'' = t^{m-2}\sum a_n(m+n)(m+n-1)t^n$. Now $P(t)y' = \frac{1}{t}\sum p_n t^n \sum a_k(m+k)t^{m+k-1}$ and so on. Thus

$$a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0, \tag{2}$$

where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \; \forall\, n \geq 0$. This equation is called the indicial equation. It can have real distinct roots, complex distinct roots, or a real (or complex) repeated root. Suppose it has two complex roots m_1, m_2. Then there are two possibilities:

1. $m_1 - m_2$ is not an integer: $f(m_2+n) \neq 0 \; \forall\, n > 0$. In fact, $|f(m_2+n)| \geq Cn^2 \; \forall\, n \geq 1$.

2. $m_1 = m_2 + n$ where n is a non-negative integer: In this case $f(m_2 + n) = 0$ at some point and hence we cannot determine all the coefficients. So we get only one formal solution y_1 and must try for a different solution using $t^{m_2} \sum b_n t^n + C \ln(t) y_1$.