1 Recap

1. Definition of connections and proof that they exist.
2. Local expression for connections and connections as vector-valued 1-forms.

2 Connections and the Levi-Civita connection

This connection matrix changes under change of trivialisation as suppose $\tilde{s}^i = [G]^i_j s^j$, then $\tilde{A} = GAG^{-1} - dGG^{-1}$. Suppose ∇_1, ∇_2 are two connections, then $\tilde{A}_1 - \tilde{A}_2 = G(A_1 - A_2)G^{-1}$, that is, the difference between two connections $\nabla_1 - \nabla_2 = d + A_1 - (d + A_2) = A_1 - A_2$ is a zeroeth order operator (simply multiplying the section locally by a matrix of 1-forms) which changes exactly like how an element of $End(E) \otimes T^*M$ ought to! That is, all connections are of the form $\nabla_0 + a$ where ∇_0 is a fixed one and a is a smooth section of $T^*M \otimes End(E)$.

Remark: For a line bundle, $\tilde{A} = A - d \ln G$. This is rather curious because $d\tilde{A} = dA$, i.e., dA defines a globally-defined closed 2-form! What is even more curious is this: If we choose another connection, then $dA = dA + da$ where a is a global function, i.e., the de Rham cohomology class of dA is independent of the connection chosen (and is in fact an invariant of the line bundle itself). We can try to play the same game for general rank vector bundles - $d\tilde{A} = d(GAG^{-1}) - d(dGG^{-1}) = dGAG^{-1} + GdAG^{-1} - GAG^{-1}dGG^{-1} - dGG^{-1}dGG^{-1}$. Ideally, if $d\tilde{A}$ had been just $GdAG^{-1}$, we could have tried looking at the *characteristic polynomial*, which would then give us globally defined $2k$-forms! To get rid of the other terms, note that if $A = 0$, then $d\tilde{A} = -(dGG^{-1}) \wedge (dGG^{-1}) = -\tilde{A} \wedge \tilde{A}$. In other words, let's try $F = dA + A \wedge A$ and see how it transforms. It is locally a matrix of 2-forms. In fact, $\tilde{F} = GFG^{-1}$, i.e., it is a section of $End(E) \otimes \Lambda^2 M$. It is called the *curvature* of the connection. (Why the word curvature? we shall see soon.) Now the characteristic polynomial of the curvature does give us globally defined even forms. It turns out these forms are closed and their cohomology classes are independent of the connection chosen. In fact it turns out that all the $4k + 2$ forms are 0 in cohomology! The others are called (upto normalisation by factors of 2π) Pontryagin classes of E, and are closely related to these things called Chern classes. These things are invariants of the vector bundle and help in classifying vector bundles. In algebraic geometry they play an even more starring role (indeed, one formulation of the famous Hodge conjecture involves these beasts).

Just like if we have a metric on E, we have induced metrics on the dual and tensor powers, a similar property holds for connections. If ∇ is a connection on E, then here is a way to define one on its dual: $d(\alpha(s)) = \nabla^*\alpha(s) + \alpha(\nabla s)$, i.e., $\nabla^*\alpha(s) := d(\alpha(s)) - \alpha(\nabla(s))$. This is a connection. Firstly, we have the following claim (how to prove it?).
Claim: Suppose T is a map taking smooth sections s to smooth functions $T(s) : M \to \mathbb{R}$ such that $T(f_1 s_1 + f_2 s_2) = f_1 T(s_1) + f_2 T(s_2)$ for smooth functions f_1, f_2, then there exists a unique smooth section α of E^* such that $T(s) = \alpha(s)$.
Using this claim, we can show that indeed $\nabla^*\alpha$ is a genuine section of E^*. In fact the

proof follows from the proof of the other properties: Indeed, $\nabla^*(f\alpha)(s) = d(f\alpha(s)) - f\alpha(\nabla s) = df\alpha(s) + fd(\alpha(s)) - f\alpha(\nabla s) = df\alpha(s) + f\nabla\alpha(s)$. Locally, if $\nabla s = (ds^i + [As]^i)e_i$, then $\nabla^*(\alpha_i e^i) = d\alpha_i e^i + \alpha_i \nabla^* e^i$. Now $(A^*)^i_j = (\nabla^* e^i)(e_j) = -e^i(\nabla e_j) = -e^i A^k_j e_k = -A^i_j$. That is, $[A^*] = -[A]^T$.

Likewise, if ∇_1, ∇_2 are connections on E_1, E_2, then we have a tensor product connection on $E_1 \otimes E_2$ defined as the linear extension of $\nabla_1 \otimes \nabla_2(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$ (indeed, every section can be locally written as linear combination of decomposable ones, and a connection's behaviour is local). That this is a genuine connection is a HW problem.

2.1 Parallel transport along piecewise smooth paths

Now we can define parallel transport. Let $\gamma : [a, b] \to M$ be a piecewise smooth path. We say that a continuous section s on γ, which is smooth at smooth points (locally extendable smoothly), is parallel along γ if $\frac{Ds}{dt} = \nabla_{\gamma'(t)} s = 0$. (Why is this well-defined? That is, why does it not depend on the smooth extension of s locally?)

Theorem 1. *Given any vector $v \in E_p$ at a point p such that $\gamma(a) = p$, we claim that there exists a continuous piecewise smooth section s along γ with $s(a) = v$ such that s is parallel.*

Proof. Cover γ with open sets which are simultaneously coordinate charts for the manifold and trivialising open sets for the bundle. We can divide the interval $[a, b]$ into subintervals I_k of equal size $\frac{b-a}{N}$ such that $\gamma[I_k)$ is in one of these open sets. We shall prove by induction on K that if the parallel section is defined on $\cup_{k=1}^{K} I_k$, then it is so on I_{K+1}. Indeed, locally on I_{K+1}, we have the following system of ODE:

$$\frac{d\vec{s}}{dt} + [A(\gamma')]\vec{s} = 0$$
$$\vec{s}(t_0) = \vec{s}_0. \tag{1}$$

By ODE theory, there exists a smooth solution of this equation on a maximal interval $[t_0, M)$. We shall prove that in fact it exists on I_{k+1}. Suppose we prove that $\vec{s}'(t)$ is uniformly bounded on $[t_0, M)$. Note that $|\vec{s}(s) - \vec{s}(t)| \leq C|s - t|$ and hence $\lim_{t \to M-} \vec{s}$ exists. Now starting at that limit and solving backwards, we see by uniqueness that this solution can be extended smoothly across M. Hence we are done. To be continued..... ∎