1 Recap

1. Connection matrix and its change under trivialisation changes.
2. Difference of connections is a section of $T^*M \otimes End(E)$. (Optional remark on Chern classes.)
3. Induced connection on duals and tensor powers.
4. Parallel transport along smooth paths. Reduction to solving an ODE. Further reduction to proving a derivative estimate.

1.1 Parallel transport along piecewise smooth paths

Theorem 1. *Given any vector $v \in E_p$ at a point p such that $\gamma(a) = p$, we claim that there exists a continuous piecewise smooth section s along γ with $s(a) = v$ such that s is parallel.*

Proof. Firstly note that $\vec{s}(t) \neq 0$ because if it is zero somewhere, then by ODE theory, it is zero throughout. Thus $u(t) = |\vec{s}(t)|$ is smooth. Now $u(t) \leq C(1 + \int t_0{}^t u(s)ds)$ and hence $u(t) \leq Ce^{C(t-t_0)}$ (why?). This means $u(t)$ is uniformly bounded on $[t_0, M)$. Note that $|\vec{s}(s) - \vec{s}(t)| \leq C|s - t|$ and hence $\lim_{t\to M-} \vec{s}$ exists. Now starting at that limit and solving backwards, we see by uniqueness that this solution can be extended smoothly across M. Hence we are done. ☐

Taking cue from curves and surfaces, what happens if we parallel transport along an infinitesimal rectangle? Can we somehow get a concept of "curvature"? If we move from p to $p+dtX$, then $ds = -[A_p(X)]sdt$. If we move $s+ds$ from $p+dtX$ to $p+dtX+daY$, then we get $(1 - [A_{p+dtX}(Y)]da)(s + ds) - s = (1 - A_{p+dtX}(Y)da)(1 - A_p(X)dt)s - s = first\ order + A_p(X) \wedge A_p(Y)s - \frac{\partial A}{\partial x^i}(Y)X^i sdadt$. Likewise if we come back and so on, we can see that the difference is proportional to $(dA + A \wedge A)(X, Y)$. This quantity $dA + A \wedge A$ is locally a matrix of 2-forms, and as we saw earlier, it is actually a section of $\Lambda^2 M \otimes End(E)$. This is called the curvature of the connection ∇.

Returning back to connections, suppose E has a metric h. Then we would want parallel transport to ideally preserve inner products, i.e., if s_1, s_2 are parallel, then $\langle s_1, s_2\rangle$ must be a constant. For this to happen, $0 = \frac{dh(s_1,s_2)}{dt} = \frac{dh_{ij}}{dt}(s_1)^i(s_2)^j - h_{ij}[A(\gamma')s_1]^i(s_2)^j - h_{ij}s_1^i[A(\gamma')s_2]^j$. Since this happens for any curve and any sections, $dh_{ij} = h_{ij}A^i_k + h_{ij}A^j_k$ (why?). Thus, in general, for any two sections and any curve $d\langle s_1, s_2\rangle = \langle\nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$. Any connection satisfying this property is said to be metric compatible with h. Given h, we claim that metric compatible connections exist (*HW*).

1.2 The Levi-Civita connection

Now we need to generalise the notion of "tangential acceleration/derivative" from Euclidean surfaces to general Riemannian manifolds, i.e., we want a connection ∇ on TM given a Riemannian metric. Obviously, we would want ∇ to be compatible with the metric (indeed, in the example of surfaces, parallel transport does preserve inner products). We would also hope that $\nabla_{\gamma'}\gamma' = 0$ is precisely the geodesic equation, i.e., that $(\gamma^i)'' + (\gamma^i)'(\gamma^j)'\nabla_{\partial_j}\partial_i = 0$ is the same as the geodesic equation. In other words, we want the connection coefficients to be the Christoffel symbols. To this end, what symmetries do the Christoffel symbols satisfy? Recalling that $\Gamma^i_{jk} = \frac{1}{2}g^{kl}\left(g_{il,j} + g_{lj,i} - g_{ij,l}\right)$, we see that if we interchange j, k the symbols are symmetric. This means that $\nabla_{\partial_j}\partial_k = \nabla_{\partial_k}\partial_j$ which implies that $\nabla_X Y - \nabla_Y X = [X, Y]$ (why?) This property is called being torsion-free (the tensor $\nabla_X Y - \nabla_Y X - [X, Y]$ is called the torsion tensor of the connection - why is this a tensor in the first place?). Here is the fundamental theorem of Riemannian geometry:

Theorem 2. *Let (M, g) be a Riemannian manifold. There is a unique connection (called the Levi-Civita connection of g) ∇ on TM that is metric compatible and torsion-free. Moreover, if $\Gamma^i_{jk}\partial_i = \nabla_{\partial_j}\partial_k$, then $\Gamma^i_{jk} = \frac{1}{2}g^{kl}\left(g_{il,j} + g_{lj,i} - g_{ij,l}\right)$.*

Proof. Let ∇ be such a connection. Then $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ by metric compatibility. Now $Z(g(X, Y)) = g(\nabla_X Z + [Z, X], Y) + g(X, \nabla_Y Z + [Z, Y])$ by torsion-freeness. Applying metric compatibility again, $Z(g(X, Y)) = g([Z, X], Y) + g(X, [Z, Y]) + X(g(Z, Y)) + Y(g(X, Z)) - g(Z, \nabla_X Y) - g(Z, \nabla_Y X)$. Now we use torsion-freeness again:

$$Z(g(X,Y)) = g([Z,X],Y) + g(X,[Z,Y]) + X(g(Z,Y)) + Y(g(X,Z)) - g(Z,[Y,X]) - 2g(Z,\nabla_X Y).$$

That is,

$$g(Z,\nabla_X Y) = \frac{1}{2}\left(g([Z,X],Y) + g(X,[Z,Y]) + X(g(Z,Y)) + Y(g(X,Z)) - g(Z,[Y,X]) - Z(g(X,Y))\right).$$

So such a connection is unique (and easily is seen to satisfy the given formula for Christoffel symbols). That this expression defines a genuine metric-compatible torsion-free connection is an exercise (*HW*). □

Here is an important "naturality" property of the LC connection. Before stating this result, note that if $\phi : M \to N$ is a diffeomorphism, and if X is a smooth vector field on M, then $\phi_* X(p) := (\phi_*)_{\phi^{-1}(p)} X_{\phi^{-1}(p)}$ is a smooth vector field on N that is ϕ-related to X.

Lemma 1.1. *If (M, g), (N, h) are Riemannian manifolds with LC connections ∇ and $\tilde{\nabla}$ respectively, and $\phi : M \to N$ is a local isometry, then $\phi_*(\nabla_X Y) = \tilde{\nabla}_{\phi_* X}\phi_* Y$ where the right-hand-side is interpreted locally.*

Proof. Indeed, suppose Z is a vector field on N. Then in a neighbourhood U (such that $\phi : \phi^{-1}(U) \to U$ is a diffeo) of a point p, $Z = \phi_* Y$ for some smooth locally defined vector field on $\phi^{-1}(U) \subset M$. Likewise, if A is a local vector field on U, $A = \phi_* X$. Define $\tilde{\nabla}_A Z := \phi_*(\nabla_X Y)$. We claim that this definition is well-defined, is metric-compatible, and torsion-free. Hence it is the LC connection of h. (*HW*) □

Given the Levi-Civita connection, there are induced connections on T^*M and the tensor bundles as before. For completeness, here is how the induced connection formula works for $(1,1)$-tensors for instance: $(\nabla_k T)^{i_j} = T^i_{j,k} + \Gamma^i_{kl}T^l_j - \Gamma^l_{kj}T^i_l$.