1 Recap

1. Parallel transport along piecewise smooth paths.
2. Curvature via parallel transport.
3. Metric compatible connections
4. Levi-Civita connection and the fundamental theorem of RG.
5. Naturality.

2 The Levi-Civita connection

Remark: Here is another way to motivate the torsion-free condition. Consider the induced connection on 1-forms. Note that $\nabla\omega$ is a 2-tensor. So $Alt(\nabla\omega)$ is a 2-form. But there is already a differential operator taking one-forms to two-forms: d. When is $Alt(\nabla\omega) = d\omega$? That is, $\omega_{i,j} - \omega_{j,i} - (\Gamma_{ij}^k - \Gamma_{ji}^k)\omega_k = \omega_{i,j} - \omega_{j,i}$. Thus $\Gamma_{ij}^k = \Gamma_{ji}^k$ which is precisely the torsion-free condition.

Now we define the classical operators of multivariable calculus. As mentioned earlier, $gradf = \nabla f = (df)^{\#}$. We can define the divergence of a vector field as $divX = tr(\nabla X) = (\nabla X)_i^i$. Thus $\Delta f = div(gradf) = f_{ij}g^{ij} + g^{ij}_{,i}f_j + \Gamma_{ki}^i g^{kj} f_j$.
We now have a version of Stokes' theorem: Suppose Y is a smooth vector field on a compact oriented Riemannian manifold (without boundary) (M, g). Then $\int divY vol_g = 0$.
Proof: The point is that the following lemma holds:
Lemma: $divY vol_g = di_Y vol_g$ where $i_Y\omega$ is the "contraction operator", i.e., $i_Y\omega(v_1, \ldots, v_{n-1}) = \omega(Y, v_1, \ldots, v_{n-1})$.
Given this lemma, we are done by the usual Stokes theorem. The proof of the lemma is as follows:
$di_Y(\sqrt{\det(g)}dx^1 \wedge \ldots) = d(\sqrt{\det(g)}(Y^1dx^2 \ldots - dx^1Y^2dx^3 \ldots))$. Assume that at p, we choose normal coordinates for g. Then $d\sqrt{\det(g)}(p) = 0$ and $\sqrt{\det(g)}(p) = 1$. Moreover, $divY(p) = Y^i_{,i}(p)$. Upon calculation, we easily see that $divY vol_g = di_Y vol_g$ at p. Since p is arbitrary, we are done. □

3 Curvature of the Levi-Civita connection

We can define the Riemann curvature endomorphism of the LC connection as we did for the usual connections but we prefer to do it from scratch as follows: $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.
Why is this a tensor? That is, we claim that $R(X,Y)Z(p)$ depends only on $X(p), Y(p), Z(p)$ and not on their values nearby (including derivatives). As before, we claim that it is enough to show linearity over smooth functions (why?) Indeed this expression is of course linear over constants. Now $R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z = f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{fXY-Y(f)X-fYX}Z = f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z -$

$f\nabla_{[X,Y]}Z+Y(f)\nabla_X Z = fR(X,Y)Z$. Likewise, we can prove linearity in the other vector fields too. Thus this Riemann curvature is a tensor. It is a $(1,3)$ tensor (because it takes in three vectors and spits out one vector) and $R(X,Y)Z = -R(Y,X)Z$. The Riemann curvature tensor is $Riem(X,Y,Z,W) = g(R(X,Y)Z,W)$. Locally, $R(\partial_j, \partial_k)\partial_l = R_{jkl}{}^i \partial_i$ (in contrast to the curvature of bundles where we would have chosen the convention $R^i{}_{jkl}$). In other words, $R = R_{jkl}{}^i dx^j \otimes dx^k \otimes dx^l \otimes \partial_i$. Also $Riem(\partial_j, \partial_k, \partial_l, \partial_i) = R_{jkli} = R_{jkl}{}^a g_{ia}$. The explicit expression for the Riemann curvature is as follows:
$R_{ijk}{}^l = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^l_{ik}\Gamma^s_{jk} - \Gamma^t_{ik}\Gamma^l_{jt}$. Clearly, for Euclidean space, this curvature is 0. It is also zero for the flat torus (that is, the metric induced on the torus from Euclidean space via Riemannian covering).

3.1 Symmetries of the Riemann curvature

Just as for any tensor, we need to understand the symmetries (under interchanges) for the Riemann curvature as well.

Theorem 1. *Let X, Y, Z, W be smooth vector fields. (Since we are dealing with tensors, the values depend only on $X(p), Y(p), Z(p), W(p)$. Thus without loss of generality, we may assume that these are coordinate vector fields.) Then*

1. $R(X,Y)Z = -R(Y,X)Z$ *($R_{ijkl} = -R_{jikl}$)*
2. $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$ *($R_{ijkl} = -R_{ijlk}$)*
3. *(First/Algebraic Bianchi identity)* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$. *($R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$)*
4. $Riem(X,Y,Z,W) = Riem(Z,W,X,Y)$. *($R_{ijkl} = R_{klij}$)*

Proof. 1. $\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z$ is obviously skew-symmetric and X, Y.

2. It is enough to prove (by multilinearity) that $Riem(X,Y,Z,Z) = 0$.

$$\begin{aligned} \langle \nabla_X \nabla_Y Z, Z\rangle &= X(\langle \nabla_Y Z, Z\rangle) - \langle \nabla_Y Z, \nabla_X Z\rangle \\ &= X(Y(\langle Z, Z\rangle)) - X(\langle Z, \nabla_Y Z) - \langle \nabla_Y Z, \nabla_X Z\rangle \\ &= Y(X(\langle Z, Z\rangle)) - \langle \nabla_Y Z, \nabla_X Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle - \langle Z, \nabla_X \nabla_Y Z\rangle \\ &= 2Y(\langle \nabla_X Z, Z\rangle) - 2\langle \nabla_Y Z, \nabla_X Z\rangle - \langle Z, \nabla_X \nabla_Y Z\rangle \\ &= 2\langle \nabla_Y \nabla_X Z, Z\rangle - -\langle Z, \nabla_X \nabla_Y Z\rangle. \end{aligned} \tag{1}$$

Thus we are done.

3. $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z$. Cyclic addition is easily shown to be zero.

4. $R_{ijkl} = -R_{jkil} - R_{kijl} = R_{jkli} + R_{kilj} = -R_{klji} - R_{ljki} - R_{ilkj} - R_{lkij} = 2R_{klij} - R_{jilk}$ and hence we are done.

□

The Riemann tensor is a rather complicated object. Using the linear algebraic operation of trace, one can simplify the notion of curvature (at a cost of potential loss of information).

1. Ricci curvature: $Ricc(Y,Z) = Tr(X \to R(X,Y)Z)$, i.e., $Ricc_{ij} = R_{kij}{}^{k}$. Note that $Ricc_{ji} = R_{kji}{}^{k} = -R_{jik}{}^{k} - R_{ikj}{}^{k} = Ricc_{ij}$ (why?). Thus the Ricci tensor is a symmetric $(0,2)$ tensor just as the Riemannian metric itself. (Interesting question: Are there Riemannian metrics such that $Ricc = \lambda g$ where λ is a constant? There are sometimes, and not, some other times. Such metrics are called Einstein metrics (because the Lorentzian version of the question models gravity in a vaccuum filled with dark energy). This question is very popular in Riemannian geometry (or at least used to be).)

2. Scalar curvature: We can take a further trace of the Ricci: $S = Ricc_{ij}g^{ij}$ to get a scalar-valued function. While we may lose information, a natural question is to ask whether there are metrics of constant scalar curvature. In fact, we can attempt to produce one by taking a conformal change $S(e^{f}g_0) = c$. So we can ask whether there are metrics of constant scalar curvature in every conformal class. This is called the Yamabe problem (and it has been solved completely).