1 Recap

1. Riemann curvature tensor and its symmetries.
2. Ricci tensor and scalar curvature.

2 Curvature of the Levi-Civita connection and Riemann's theorem

We have the following crucial naturality principle for the Riemann curvature tensor (why?):
If $\phi : (N, h) \to (M, g)$ is a smooth immersion such that $h = \phi^* g$, then $\phi_*((R_h(X,Y)Z)_p) = R_g(\phi(p))(\phi_* X, \phi_* Y)\phi_* Z$ and $Riem_h = \phi^* Riem_g$.
We can now prove Riemann's theorem:

Theorem 1. *There exist local coordinates in a neighbourhood U of p where g is Euclidean iff $Riem_g \equiv 0$ in a neighbourhood V of p.*

Proof. If g is Euclidean, obviously $Riem_g \equiv 0$. So we shall assume that $Riem_g \equiv 0$. The key point is that suppose we manage to find an orthonormal basis E_i (in some neighbourhood) that is parallel then $[E_i, E_j] = \nabla_{E_j} E_i - \nabla_{E_i} E_j = 0$. Therefore by Frobenius's theorem (that we discussed in the first few lectures), we can find a coordinate chart such that $E_i = \partial_i$. Thus we are done.
Here is a proof of the existence of such an orthonormal basis: Consider a coordinate neighbourhood and consider a coordinate rectangle $[0,a] \times [0,a] \ldots$. First consider any orthonormal basis at the origin and parallel transport it along the x^1 axis to get a parallel collection of vector fields that form an orthonormal basis on $[0,a] \times 0 \times 0 \ldots$. (This basis is smooth in x^1 because of the smooth dependence of parameters of solutions of ODE). Now parallel transport from $(t, 0, \ldots)$ to $(t, x^2, 0, \ldots)$ and so on. We then get a smooth collection of vector fields that form an orthonormal basis everywhere. Note that $\nabla_{\partial_1} E_i = 0$ on $[0,a] \times 0 \times \ldots$. We claim that $\nabla_{\partial_1} E_i = 0$ on $[0,a] \times [0,a] \times 0 \times \ldots$ as well: Indeed, $\nabla_{\partial_2} \nabla_{\partial_1} E_i = \nabla_{\partial_1} \nabla_{\partial_2} E_i = 0$ (because the curvature vanishes). Thus $\nabla_{\partial_1} E_i$ is parallel along the x^2-axis. Since it is 0 at $[0,a] \times 0 \ldots$, it is zero throughout. (Parallel transport is unique or alternatively, it preserves lengths.) Likewise, inductively we can conclude that E_i are parallel. □

2.1 Symmetries of the Riemann curvature and other measures of curvature

We also have a differential Bianchi identity: $(\nabla_X R)(Y,Z)) + (\nabla_Z R)(X,Y) + (\nabla_Y R)(Z,X)) = 0$ where ∇ acts on R treating it as a $(3,1)$ tensor. Thus without loss of generality, assume X, Y, Z are coordinate vector fields in normal coordinates at the point.

$$(\nabla_X R)(Y,Z)W = \nabla_X (R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W. \quad (1)$$

Since the coordinates are normal, $\nabla_X Z = 0$ etc. Hence at p,

$$(\nabla_X R)(Y, Z)W = \nabla_X \nabla_Y \nabla_Z W - \nabla_X \nabla_Z \nabla_Y W. \tag{2}$$

Cyclic addition easily produces zero. $\square$

Sectional curvature: Given a two-plane spanned by X and Y, we can define $K(two-plane) = \frac{\langle R(X,Y)Y,X\rangle}{|X|^2|Y|^2-\langle X,Y\rangle^2}$ (why is this independent of X and Y and dependent only on the two-plane). Note that if X and Y are orthonormal, it is simply $Riem(X, Y, Y, X)$. So we have a number associated to every two-plane. The point is that the sectional curvatures completely determine the Riemann tensor. Indeed, (Exercise) $6Riem(X, Y, Z, W)$ is the coefficient of t^2 in

$$\begin{aligned} &Riem(X + tW, Y + tZ, Y + tZ, X + tW) - t^2 Riem(X, Z, Z, X) - t^2 Riem(W, Y, Y, W) \\ &-Riem(Y + tW, X + tZ, X + tZ, Y + tW) + t^2 Riem(Y, Z, Z, Y) + t^2 Riem(W, X, X, W). \end{aligned} \tag{3}$$

We can now define the notion of "constant curvature manifold" as simply a Riemannian manifold where the sectional curvatures are the same for all points and all planes. We can also talk about "positively curved manifolds" as sectional curvature at all points for all planes being positive and so on. As we shall see later, there aren't many simply connected manifolds with constant sectional curvature.

2.2 Curvature of model spaces (the space forms)

$\mathbb{R}^n$ has curvature zero. Are there manifolds with constant sectional curvatures? There are. Consider S^n, g_{S^n} with radius R and $\mathbb{H}^n, g_{\mathbb{H}^n} = \frac{R^2 \sum_i dx^i \otimes dx^i}{(x^n)^2}$.

A few facts can help us calculate the curvature (in general):

1. Curvature in normal coordinates: (HW) $R_{ijkl}(P) = \frac{1}{2}\left(g_{ik,jl} + g_{jl,ik} - g_{il,jk} - g_{jk,il}\right)(P)$.

2. Curvature for a conformal change: (HW) If $\tilde{g} = e^{2\phi} g$, then $\tilde{R}_{ijkl} = e^{2\phi} R_{ijkl} - e^{2\phi}(g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il})$ where $T_{ij} = \nabla_i \nabla_j \phi - \nabla_i \phi \nabla_j \phi + \frac{1}{2}|d\phi|^2 g_{ij}$.

3. Isometry groups act transitively for model spaces:

   (a) Euclidean space: Obviously translations act transitively.

   (b) Sphere: Consider $O(n + 1)$. It acts on the sphere and since it is an isometry of $\mathbb{R}^{n+1}$, it acts isometrically on the sphere. Now we claim that any point P can be taken to the north pole N via an element of $O(n + 1)$. Indeed, we prove it by induction. For S^1, this is easily true. Assume it is true for S^{n-1}. Now consider any hyperplane containing the origin, P, and N. Choose any orthonormal basis for this plane and extend it by producing a normal vector. The intersection of this plane with S^n is S^{n-1}. Use the induction hypothesis to produce an $O(n)$ matrix taking P to N within this plane. Now extend it by fixing the normal vector in $\mathbb{R}^{n+1}$.

(c) Hyperbolic space: We can prove that (HW)$\mathbb{H}^n$ is isometric to two other models, the Poincaré ball model $\mathbb{B}^n$, $g^n_{\mathbb{B}} = \frac{4R^2}{(R^2-|x|^2)^2} g_{Euc}$, and the hyperboloid model (hence hyperbolic metric) $\mathbb{H}yp$ which is the set of points on the upper branch of the hyperboloid $Q(x) = \sum_i (x^i)^2 - (x^{n+1})^2 = -R^2$ equipped with the induced bilinear form from Q. It turns out that despite Q being Lorentzian, the induced bilinear form is a Riemannian metric. The group $SO^+(n,1)$ preserves the bilinear form and hence the metric. A similar proof as above shows that this group also takes any point P to the "pole" $(0, 0 \ldots, 0, 1)$. (Basically using $O(n)$, take the first n coordinates to $(0, 0, \ldots, a)$ and then use an appropriate Lorentz transformation.)