1 Recap

1. Riemann's theorem.
2. Sectional curvature.
3. Isometries of model spaces.

1.1 Curvature of model spaces (the space forms)

Given these facts, we have the following formula.

$$R_{ijkl} = \lambda(g_{il}g_{jk} - g_{ik}g_{jl}), \tag{1}$$

where $\lambda = 0$ for Euclidean space, R^{-2} for the sphere and $-R^{-2}$ for hyperbolic space. In particular, to calculate the sectional curvature of a plane, assume wLOG that we have normal coordinates such that ∂_i, ∂_j span the plane. Then $R_{ijji} = \lambda$. Hence these spaces have constant sectional curvature and these metrics are also Einstein metrics.

Proof. By rescaling, assume that $R = 1$. For hyperbolic space, use the formula for conformal change to directly calculate. For the sphere, we can calculate only at the north pole by choosing $x^{n+1} = \sqrt{1-(x^1)^2 \ldots}$. It turns out these coordinates are normal at that point. Hence we use the formula in normal coordinates. □

It is a rather intriguing question to know whether these are the only constant sectional curvature spaces. Of course not! The torus is another example. However, are these the only simply connected ones? (That is, is the universal cover of every constant sectional curvature space one of these "space forms"?) The answer is yes! (The Killing-Hopf theorem). We shall prove this later. In fact, if the sectional curvature is ≤ 0, the universal cover is diffeomorphic to $\mathbb{R}^n$ (Cartan-Hadamard theorem). The Killing-Hopf theorem combined with the observation that the Ricci tensor determines Riemann in 3 dimensions implies that to prove the Poincaré conjecture, it is enough to produce an Einstein metric on every compact simply connected manifold. Hamilton's Ricci flow was designed specifically for this purpose. In fact, Hamilton proved that if there is a metric with positive sectional curvature on a compact simply connected manifold, then the manifold is a three-sphere. To remove this assumption required a lot of ingenious work, which was done by Perelman.
We can also compute geodesics in these model spaces.

1. Euclidean space: One can easily see that straight lines are the geodesics (indeed $\gamma''(t) \equiv 0$).
2. Sphere: Using the naturality properties of the Christoffel symbols, one can see that it is enough to consider geodesics passing through the north pole. One can verify that any plane passing through the origin and the north pole intersects the sphere in a geodesic. (One can verify this in graph coordinates.) So all these geodesics intersect at the south pole.

3. Hyperbolic space: Using naturality properties, one can prove that it is enough to consider $(0, 0, \ldots, 1)$ in $\mathbb{H}^n$. Now one can verify that vertical lines and arcs of circles with centre on $x^{n+1} = 0$ are the only geodesics (indeed they are geodesics and every tangent vector is attained as a starting vector).

As a consequence, all geodesics on these spaces are defined on $(-\infty, \infty)$. Spaces where a geodesic through every point in every direction is defined on $(-\infty, \infty)$ are called geodesically complete. Note that $\mathbb{R}^2 - (0, 0)$ with the Euclidean metric is not geodesically complete. It turns out (Hopf-Rinow) that geodesic completeness is equivalent to metric space completeness. (By the way, a similar property does not make sense for Lorentzian manifolds and is the starting point of the complications of the singularity theorems in general relativity. In fact, one way to define a singularity is through geodesic incompleteness.)

2 Geometry of submanifolds

Recall that for surfaces in $\mathbb{R}^3$, we defined the Gaussian curvature using the shape operator. What is the relationship between this curvature and say, the scalar curvature of the induced metric? How can this be generalised to arbitrary Riemannian submanifolds of Riemannian manifolds?

To this end, let $(M, g) \subset (\tilde{M}, \tilde{g})$ be a Riemannian submanifold. The first point is that $T\tilde{M}$ restricts to M as a vector bundle (why?) Now at every point, $T_p\tilde{M} = T_pM \oplus N_pM$ where $N_pM = T_pM^{\perp}$. This splitting actually is a vector bundle splitting $T\tilde{M} = TM \oplus NM$ (why?) This NM is called the normal bundle of M. Note that if M is a regular level set, NM is trivial (why?) It turns out that if $dim(M) = n-1, n-2$, then the converse holds but it is not known whether it holds in general.

The first observation is this: The Levi-Civita connection $\nabla_X Y$ for (M, g) is obtained from that of $\tilde{M}, \tilde{g}$ by the formula $\nabla_X Y = (\tilde{\nabla}_X Y)^T$. Indeed, define $D_X Y = (\tilde{\nabla}_X Y)^T$. This is a connection (why?). Now

$$g(D_X Y, Z) = \tilde{g}(\tilde{\nabla}_X Y - \sum_i \tilde{g}(\tilde{\nabla}_X Y, N_i)N_i, Z) = \tilde{g}(\tilde{\nabla}_X Y, Z) = X(g(Y, Z)) - \tilde{g}(Y, \tilde{\nabla}_X Z.$$

Hence it is metric-compatible. Likewise, it is torsion-free. Hence it is the Levi-Civita connection.

So what is $(\tilde{\nabla}_X Y)^{\perp}$ then? It actually depends on only $X(p), Y(p)$ (why?) and is a symmetric $(1, 2)$-tensor $\Pi(X, Y) = (\tilde{\nabla}_X Y)^{\perp} = \tilde{\nabla}_X Y - \nabla_X Y$ (why?) called the second fundamental form. Let X, Y, Z, W be vector fields on M and let ν be a section of the normal bundle. Then we have: The Weingarten equation $\langle \tilde{\nabla}_X \nu, Y \rangle = -\langle \nu, \Pi(X, Y) \rangle$ (thus confirming that this is the same second fundamental form as in the case of surfaces): Indeed,

$$\langle \tilde{\nabla}_X \nu, Y \rangle = X\langle \nu, Y \rangle - \langle \nu, \tilde{\nabla}_X Y \rangle = 0 - \langle \nu, (\tilde{\nabla}_X Y)^{\perp} \rangle = -\langle \nu, \Pi(X, Y) \rangle.$$