1 Recap

1. Curvature of model spaces.
2. Geodesics in model spaces.
3. Normal bundle, second fundamental form, and the Weingarten equation.

2 Geometry of submanifolds

$\Pi(X,Y) = (\tilde{\nabla}_X Y)^{\perp} = \tilde{\nabla}_X Y - \nabla_X Y$ (why?) called the second fundamental form. Let X, Y, Z, W be vector fields on M and let ν be a section of the normal bundle. Then we have

The Gauss-Codazzi equation $Rm(X,Y,Z,W) = \tilde{Rm}(X,Y,Z,W) + \langle \Pi(X,W), \Pi(Y,Z)\rangle - \langle \Pi(X,Z), \Pi(Y,W)\rangle$ (thus scalar is upto a factor, the Gauss curvature): Indeed (we use Weingarten below), assuming wlog that X, Y, Z, W are coordinate vector fields,

$$\begin{aligned}
\langle \tilde{\nabla}_X \tilde{\nabla}_Y Z, W\rangle &= \langle \tilde{\nabla}_X (\nabla_Y Z + \Pi(Y,Z)), W\rangle \\
&= \langle \tilde{\nabla}_X \nabla_Y Z, W\rangle - \langle \Pi(Y,Z), \Pi(X,W)\rangle \\
&= \langle \nabla_X \nabla_Y Z, W\rangle + \langle \Pi(X, \nabla_Y Z), W\rangle - \langle \Pi(Y,Z), \Pi(X,W)\rangle \\
&= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \Pi(Y,Z), \Pi(X,W)\rangle. \qquad (1)
\end{aligned}$$

Likewise, we have a formula with X, Y interchanged. Upon subtraction, we arrive at the desired result.

Note that as a special case, if $\tilde{Rm} = 0$, then in particular for the sectional curvature, $K(X,Y) = \frac{\langle \Pi(X,W), \Pi(Y,Z)\rangle - \langle \Pi(X,Z), \Pi(Y,W)\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$, i.e., the ratio of the determinants of the two fundamental forms. That is the Gaussian curvature is the sectional curvature.

3 Local behaviour of geodesics

We recall that an admissible curve is a continuous, piecewise smooth, piecewise regular (meaning it is either constant on a piece or the velocity is non-zero on that piece) map from $[a,b]$ to M. An admissible endpoints fixed-variation $\Gamma(t,s)$ around γ is a continuous map from $[a,b] \times [c,d]$ (where $0 \in [c,d]$) to M such that $\Gamma(t,0) = \gamma(t)$, $\Gamma(b,s) = \gamma(b)$, $\Gamma(a,s) = \gamma(a)$, and Γ is smooth on $[a_i, a_{i+1}] \times [c,d]$. The variation vector field along γ, i.e., V is $\frac{\partial \Gamma}{\partial s}|_{s=0}$. We note (due to the HW) that at every smooth point t_0, there is a neighbourhood U of $\gamma(t_0)$ and a smooth vector field that coincides with V on $\gamma(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

Recall that a geodesic is the same as $\nabla_{\gamma'}\gamma' = 0$. Using the theory of ODE as well as the observation that linear reparametrisations of geodesics are geodesics, we have the following result.

Theorem 1. *Given $p \in M$ and $v \in T_pM$, there exists $\epsilon_{|v|,p} > 0$ and a unique geodesic $\gamma : (-\epsilon_{|v|,p}, \epsilon_{|v|,p}) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Moreover, $\lim_{|v|\to 0} \epsilon_{|v|,p} = \infty$ and we can choose $\epsilon_{|v|,p}$ uniformly for all q in a neighbourhood of p.*

For every geodesic, there is an open interval I that is the maximal interval of existence. Also note that the speed of γ is constant throughout the geodesic (because parallel transport preserves lengths).
Here is an example of geodesics on a cylinder $x^2 + y^2 = 1$: A general curve is $(\cos(\theta(t)), \sin(\theta(t)), z(t))$. The geodesic equation is

$$\begin{aligned} -\theta'' \sin(\theta) &= (\lambda(t) + (\theta')^2) \cos(\theta) \\ \theta'' \cos(\theta) &= (\lambda(t) + (\theta')^2) \sin(\theta) \\ z''(t) &= 0. \end{aligned} \tag{2}$$

Thus $z(t) = z + 0 + bt$ and $\theta(t) = at + \theta_0$.
The same calculations as earlier can be used to prove the following first variation formula.

Theorem 2. *Let $\gamma : [a, b] \to M$ be an admissible curve, Γ an admissible variation, and V the variation vector field. Then*

$$\frac{dE}{ds}|_{s=0} = \langle V(b), \gamma'(b-)\rangle - \langle V(a), \gamma'(a+)\rangle - \sum_{k=1}^{m-1} \langle V(a_k), \Delta_k \gamma'\rangle - \int_a^b \langle V, D_t \gamma'\rangle. \tag{3}$$

If $|\gamma'(t)| = 1$, then $\frac{dL}{ds}|_{s=0} = \frac{dE}{ds}|_{s=0}$.

In fact, this proof can be simplified by noting the following useful commutativity property (which is easy to prove).

$$\frac{D}{ds}\frac{\partial \Gamma}{\partial t} = \frac{D}{dt}\frac{\partial \Gamma}{\partial s}. \tag{4}$$

We now prove the following theorem.

Theorem 3. *An admissible curve is a critical point of E w.r.t variations that fix the endpoints iff it is a smooth geodesic.*
Moreover, an admissible curve is a critical point of L iff there is a reparametrisation that makes it into a smooth geodesic.

Note that since geodesics preserve speed, and L is reparametrisation invariant, we can assume that it is arc-length parametrised and hence the critical points of length and energy coincide. Unit speed length minimisers are thus geodesics. (The other way round is of course false! (why?))

Proof. Firstly, on (a_{k-1}, a_k), we claim that $\frac{D}{dt}\gamma' = 0$. Indeed, let $t_0 \in (a_{k-1}, a_k)$. Consider $I_0 = [t_0 - \delta, t_0 + \delta]$ such that $\gamma(I_0)$ is inside a coordinate chart such that $\gamma(t) = (t, 0, \ldots, 0)$ by using the constant rank theorem. The vector field $\tilde{V} = \chi \frac{D}{dt}\gamma'$ is along $\gamma(t)$ where χ is a bump function in t centred at t_0 and supported in I_0 (so this vector field is zero outside I_0). Consider $\Gamma(t, s) = \gamma(t) + s\tilde{V}(t)$ when $t \in I_0$ and $\Gamma(t, s) = \gamma(t)$ otherwise. Note that Γ is an admissible variation that fixes the endpoints (why?) Also the variation vector field is $\tilde{V}(t)$. By the first variation formula, we are done.
We can construct a variation such that $V(a_k) = \Delta_k \gamma'$ and $V = 0$ at other problematic points. By the first variation formula, again we see that the energy can be made to strictly decrease if $\Delta_k \gamma' \neq 0$.
By uniqueness of geodesics, we see that γ is smooth. $\square$

3.1 Exponential map

Note that at any point, there is for all small enough tangent vectors, there are geodesics through all directions for a uniform time. Following the geodesics for some time, we may hope that all nearby points are reached. More precisely, using rescaling of the parameter we see that for small enough tangent vectors there are geodesics for time 1. Define $\exp(p, v) = \gamma_v(1)$ where $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. The domain of this exponential map consists of elements of TM whose corresponding geodesics exist for at least unit time. (This domain contains an open ball around the origin.)

We have the following result.

Theorem 4.

1. *The domain $\mathcal{E}$ of the exponential map is an open subset of TM and $\exp$ is a smooth map.*
2. *For all $v \in T_pM$, $\gamma_v(t) = \exp(p, tv)$ if $(p, tv) \in \mathcal{E}$. Moreover, $\frac{d}{dt}|_{t=0} \exp(p, tv) = \gamma'(0)$.*
3. *$\mathcal{E}_p$ (which is $\mathcal{E} \cap \pi^{-1}(p)$) is an open subset of T_pM which is star-shaped w.r.t $\vec{0}$, i.e., $tv \in \mathcal{E}_p \; \forall 0 \leq t \leq 1$ if $v \in \mathcal{E}_p$.*
4. *The map $\phi : \mathcal{E} \to M \times M$ given by $\phi(p, v) = (p, \exp(p, v))$ is a local diffeomorphism at any point $(p, 0)$, and $\exp_p(v)$ is a local diffeomorphism from an open subset of T_pM to a neighbourhood of p in M.*

Choosing an orthonormal basis for T_pM, we get a coordinate chart around p called geodesic normal coordinates.