1 Recap

1. Gauss-Codazzi equation.

2. First variation formula and the fact that critical points of the length and energy functionals are smooth geodesics (well, after reparametrisation for length). Length can be reducing by "rounding the corner". So we can assume that the admissible curves are actually C^1, regular, and piecewise smooth.

3. Exponential map definition.

1.1 Exponential map

Theorem 1. *1. The domain $\mathcal{E}$ of the exponential map is an open subset of TM and* $\exp$ *is a smooth map.*

2. *For all $v \in T_pM$, $\gamma_v(t) = \exp(p, tv)$ if $(p, tv) \in \mathcal{E}$. Moreover, $\frac{d}{dt}|_{t=0} \exp(p, tv) = \gamma'(0)$.*

3. *$\mathcal{E}_p$ (which is $\mathcal{E} \cap \pi^{-1}(p)$) is an open subset of T_pM which is star-shaped w.r.t $\vec{0}$, i.e., $tv \in \mathcal{E}_p \, \forall 0 \leq t \leq 1$ if $v \in \mathcal{E}_p$.*

4. *The map $\phi : \mathcal{E} \to M \times M$ given by $\phi(p, v) = (p, \exp(p, v))$ is a local diffeomorphism at any point $(p, 0)$, and $\exp_p(v)$ is a local diffeomorphism from an open subset of T_pM to a neighbourhood of p in M.*

Choosing an orthonormal basis for T_pM, we get a coordinate chart around p called geodesic normal coordinates. The "radial distance" is $r = \sqrt{(x^1)^2 + \ldots}$ and the "radial vector field" is $\frac{\partial}{\partial r} := \frac{x^i}{r}\partial_i$.
Also, as a corollary of the above result, for any admissible path γ and any continuous piecewise smooth vector field V along γ, there exists an $\epsilon > 0$ and an admissible variation $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ of γ such that $V = \partial_s\Gamma(0, t)$. Indeed, define $\Gamma(s, t) = \exp(\gamma(t), sV(t))$. $\frac{\partial}{\partial s}\Gamma(0, t) = V(t)$.

Proof. 1. Given any (p, v) in the domain, the result for existence of geodesics shows that there is a uniform time of existence independent of p, v as long as $p \in U$ and $|v| < \epsilon$. Now rescale the time parameter to make sure that the time of existence is 1 (at the cost of reducing ϵ if necessary). The smooth dependence on parameters for ODE shows that $\exp$ is smooth.

2. $\gamma'_{tv}(0) = tv$ and $\gamma_{tv}(0) = p$. Suppose $\gamma'_v(0) = v$ and $\gamma_v(0) = p$. We need to show that $\gamma_{tv}(1) = \gamma_v(t)$. Now consider $\tilde{\gamma}(s) = \gamma_{tv}(\frac{s}{t})$. Then $\tilde{\gamma}$ is still a geodesic and $\tilde{\gamma}'(0) = v$, $\tilde{\gamma}(0) = p$ and hence by uniqueness $\gamma_v(t) = \tilde{\gamma}(t) = \gamma_{tv}(1)$. Differentiating w.r.t t at $t = 0$ we get the desired result.

3. The aforementioned argument shows that $\mathcal{E}_p$ is open. Now if $v \in \mathcal{E}_p$, i.e., γ_v exists on $[0, 1]$, then $\gamma_{tv}(s) = \gamma_v(ts)$ and hence it is star-shaped.

4. Note that $(\phi_*)_{p,0} = \begin{bmatrix} I & 0 \\ D_p \exp(p,0) & D_v \exp(p,0) = I \end{bmatrix}$. Thus the derivative is an isomorphism and moreover, if we fix p, then the derivative of $\exp(p,.)$ is also an isomorphism. Hence by IFT we are done.

□

For geodesic normal coordinates, we have the following result.

Theorem 2. *1.* $x^i(p) = 0$, $g_{ij}(p) = \delta_{ij}$.

2. $\gamma_v(t) = (tv^1, \dots, tv^n)$ *if* $v \in T_pM$.

3. $\gamma_v'(t) = |v|\partial_r$.

4. $\Gamma^k_{ij}(p) = 0$ *and* $\partial_k g_{ij}(p) = 0$.

Proof. 1. $\exp(0) = p$. Since $(\exp_*)_0 = Id$, and we have chosen an orthonormal basis, $g_{ij}(p) = \delta_{ij}$.

2. $\exp(tv) = \gamma_v(t)$ as proven earlier.

3. $\gamma_v'(t) = v = |v|\partial_r$.

4. $(\gamma_v'')^i + \Gamma^i_{jk}(\gamma_v(t))(\gamma_v')^i(\gamma_v')^j = 0$ and hence $0 + \Gamma^i_{jk}(tv)v^iv^j = 0$. At $t = 0$, this means that $\Gamma^i_{jk} = 0$. Thus at p, $g_{il,j} + g_{lj,i} - g_{ij,l} = 0$. Interchanging i and l and adding we see that $g_{li,j} = 0$.

□

We know want to see if geodesics are at least locally length minimising. We have the following theorem.

Theorem 3. *For any point $p \in M$, there exists a "geodesically convex neighbourhood" W, i.e., there exists $\epsilon > 0$ such that for any $x, y \in W$, there is a unique smooth geodesic of length $< \epsilon$ $\gamma : [0,1] \to W$ such that $\gamma(0) = x$, $\gamma(1) = y$. Moreover this geodesic is length-minimising.*

As a corollary, geodesics are minimising for short periods of time. Indeed, for a short enough time, the length is $< \epsilon$ and the geodesic lies in a geodesically convex neighbourhood. Hence it is minimising.
To prove such a result, let us try to see if the $\exp(tv)$ is minimising from $t_1 \in (0,1)$ to $t_1 < t_2 \in (0,1)$. Suppose $\sigma(t)$ is another admissible curve connecting these points. We use polar coordinates for the tangent space. Then $\sigma'(t) = r'(t)\partial_r + v(t)$ where $v(t)$ is a linear combination of the other basis vectors. If ∂_r is orthogonal to $v(t)$ using the inner product $g(\sigma(t))$, then $\|\sigma'\| \geq |r'(t)|\|\partial_r\|$. Since $\int |r'| \geq |\int r'(t)| = |t_2 - t_1||v|$, we see that radial geodesics that lie in a region where $\exp$ is a diffeomorphism are minimising at least among all curves that lie within that region. So we need to prove the following Gauss lemma:

Lemma 1.1. *Denote by $S_{p,\eta}$ the set of tangent vectors in T_pM of length η. Let $U_{p,\delta}$ be a geodesic ball of radius δ centred at p, that is, the diffeomorphic image of $B(0,\delta)$ under $\exp_p$. Likewise, $\partial U_{p,\delta} = \exp_p(S_{p,\eta})$.*
The radial vector field ∂_r is orthogonal to $\partial U_{p,\delta}$

Proof. Let $x_0 = \exp(v_0)$ lie on the geodesic sphere. Let $v(s)$ be a curve in the tangent space lying on the sphere such that $v(0) = v_0$. Let $X = \sigma'(0)$ where $\sigma(s) = \exp(p, v(s))$ and let $\gamma(t) = \exp(p, tv_0)$. We need to show that X is perpendicular to $\gamma'(1)$. Define $\Gamma(s,t) = \exp(tv(s))$. Now let $f(s,t) = \langle \partial_s \Gamma, \partial_t \Gamma \rangle$. Suppose we show that f is a constant in t, then $f(s,0) = 0$ (because $\exp_p$ is an isometry at the origin) and hence we will be done.
Now $\partial_t f = \langle D_t \partial_s \Gamma, \partial_t \Gamma \rangle + 0 = \langle D_s \partial_t \Gamma, \partial_t \Gamma \rangle = \frac{1}{2}\partial_s \|\partial_t \Gamma\|^2 = 0$. □

As a corollary, $\nabla r = \partial_r$ (why?), as seen earlier, among admissible curves connecting q_1 and q_2 (which are on a radial geodesic from p) lying entirely in a geodesic ball are longer in length than radial geodesics (and are the unique such curves), and by continuity of the distance function, radial geodesics are length minimising (even starting from p).
Moreover, every open geodesic ball $U_{p,\delta}$ is a metric δ-ball. Indeed, if there is a point q in the metric ball that is not in the geodesic ball, then consider a curve connecting p to q of length $< \delta$. Now that curve exists the geodesic ball at some point. The length till that point is at least δ and that is a contradiction. (This also proves that closed metric balls of sufficiently small radius are compact and that the closure of the open metric ball is the closed one.)