1 Recap

1. Gauss lemma and corollary that geodesic balls are metric balls.
2. Stated theorem on existence of geodesically convex neighbourhoods.

2 Local behaviour of geodesics

Theorem 1. *For any point $p \in M$, there exists a "geodesically convex neighbourhood" W, i.e., there exists $\epsilon > 0$ such that for any $x, y \in W$, there is a unique smooth geodesic of length $< \epsilon$ $\gamma : [0,1] \to W$ such that $\gamma(0) = x$, $\gamma(1) = y$. Moreover this geodesic is length-minimising.*

Proof of the main theorem:

1. Weak geodesic convexity: By the ODE theory (and rescaling) based results mentioned earlier, there exists an ϵ and a neighbourhood U of p such that ϕ is a diffeomorphism on $\tilde{V} = \{(x,v)\ vert\ x \in U,\ |v|_x < \epsilon\}$. Suppose $W \subset U$ is a neighbourhood such that $W \times W \in \phi(\tilde{V})$. Then given any two points q_1, q_2 in W, $q_2 = \exp_{q_1}(v)$ for $|v| < \epsilon$. Let $\gamma(t) = \exp(tv)$. This is a geodesic connecting q_1, q_2 with length $< \epsilon$. This geodesic is unique as well.
2. Minimality: We have already shown that radial geodesics are minimal in length compared to admissible curves lying entirely within the geodesic ball of radius ϵ. Let r_0 be the length of the radial geodesic connecting x, y lying in W. Now $W \subset U_{x,\epsilon}$. Let $\sigma : [0,1] \to M$ with $\sigma(0) = x$ and $\sigma(1) = y$. If σ remains in $U_{x,\epsilon}$ then $L(\sigma) \geq r_0$. If σ exists this geodesic ball, then at the first point of exit, the length till that point is at least $\epsilon \geq r_0$. Hence this geodesic is minimal in length.
3. Strong geodesic convexity: This will be a guided HW but basically, one first proves that $r \circ \gamma$ for non-radial geodesics γ contained in a geodesic ball $U_{p,\delta}$ (for sufficiently small δ) attains its max at one of the endpoints (HW). Then suppose $\delta < \epsilon/2$ (where ϵ is as above) such that $U_{p,\delta} \subset W$. Suppose $x, y \in U_{p,\delta}$. Then $y = \exp_x(v)$. The geodesic $\exp_x(tv)$ lies in $U_{p,\epsilon}$ (why?) and hence $r(\exp_x(tv)) \leq \max(r(x), r(y)) < \delta$ and hence lies in $U_{p,\delta}$.

□

3 Hopf-Rinow theorem and its consequences

We first note that $\bar{B(p, r)}$ is the closed metric ball. This is not true for general metric spaces. (Take a space X with $\rho(x, y) = 1$ iff $x \neq y$. Then $\bar{B}(x, 1) = \{x\}$ but the closed metric ball is all of X.) Indeed, $\bar{B(p, r)}$ is contained in the closed ball (the closed ball is a closed set because the distance is continuous). Suppose x is in the closed ball but not in the closure. Then $d(p, x) = r$. Thus there is a sequence of unit speed admissible curves connecting p and x such that $L(\gamma_n) = l_n < r + \frac{1}{n}$. Let $x_n = \gamma(l_n - \frac{2}{n})$. Then $d(p, x_n) \leq l_n - \frac{2}{n} < r$. So $x_n \in B_r(p)$ and hence x is a limit point.

The Hopf-Rinow theorem links geodesic completeness with metric space completeness:

Theorem 2. *TFAE.*

1. *(M, d_g) is a complete metric space.*

2. *(M, g) is geodesically complete.*

3. *For all $p \in M$, $\mathcal{E}_p$ (the domain of $\exp_p$) is all of T_pM.*

4. *Closed and bounded sets are compact.*

Moreover, if any of the equivalent conditions hold, then there is a minimal unit-speed smooth geodesic connecting any two points.

Proof. $2 \Rightarrow 3$ is trivial. We now prove $1 \Rightarrow 2$:
If $\gamma : I \to M$ is a unit-speed smooth geodesic defined on the maximal interval I, such that $I \neq \mathbb{R}$, that is, WLog, $l = \sup I < \infty$, then consider the points $x_n = \gamma(l - p_n)$ where $p_n \to 0$. Note that $d(x_n, x_m) \leq |p_n - p_m|$ and hence these form a Cauchy sequence. Since M is complete, $x_n \to x$. Now there exists a neighbourhood U of x and $\epsilon > 0$ such that for all $x \in U$ with $|v|_x < \epsilon$, $\exp(x, v)$ exists. For n large enough, $x_n \in U$ and $t_n = l - p_n > l - \epsilon$. Let $\sigma_n(t) = \exp(x_n, tv_n)$ where $v_n = \gamma'(t_n)$. This geodesic exists for $t \in (0, \epsilon)$. Now by uniqueness, $\sigma_n(t) = \gamma(t + t_n)$. Since $l - t_n < \epsilon$, $\sigma_n(s)$ exists for $s \leq l - t_n$ and hence $\gamma(t)$ exists on $[l, l + \frac{\epsilon}{2})$ which is a contradiction.
Now we shall prove that $3 \Rightarrow 5$. This is the main point. In fact, given this fact, $3 + 5 \Rightarrow 4$. Indeed, if S is a closed and bounded set, then its diameter d is finite. Let $K = \exp(\bar{B(0, d)})$. Now K is compact (being the image of a compact set under a continuous map) and $S \subset K$ because of 5. Moreover, $\Rightarrow 1$ is easy because it is a general property of metric spaces. (By the way, we can also directly prove that $3 + 5$ implies 1.)
Here is the proof that $3 \Rightarrow 5$:
In fact, we shall prove that if $\exp_x$ is defined on all of T_xM, then it can be connected to any point y by a length minimising geodesic. Let $S_{x,\epsilon}$ be a closed geodesic sphere (which is also a metric sphere). It is compact and hence there is a point $p = \exp_x(\epsilon v)$ (where $|v|_x = 1$) on it which is closest to y. The geodesic $\exp(tv)$ extends for $t \in \mathbb{R}$. Let $l = d_g(x, y)$. We claim that $\exp(x, lv) = \exp(p, (l - \epsilon)v) = y$ and that $\exp(tv)$ is minimal. Both of these can be encoded in the statement $d(\gamma_v(t), y) = l - t$. Let T be the supremum of all times such that this statement is true. Note that this supremum is a maximum (by continuity). We prove two claims.

1. $d(p, y) = l - \epsilon$ and hence $T \geq \epsilon > 0$: By the triangle inequality, $l = d(x, y) \leq d(p, x) + d(p, y) = \epsilon + d(p, y)$ and hence $d(p, y) \geq l - \epsilon$. To prove the opposite inequality, if σ is any admissible curve connecting x and y, it hits the sphere at some point p_1. Now $d(x, p_1) = \epsilon$ and $d(p_1, y) \geq d(p, y)$ and hence $l(\sigma) \geq \epsilon + d(p, y)$. Hence, $d(p, y) \leq l - \epsilon$.

2. $T = l$: Suppose $T < l$. Then let $p' = \gamma(T)$. Now $l(\gamma_v(T) = p', y) = l - T$. As before, consider a geodesic ball (which is also a metric ball) of radius ϵ' around p'. Let $p'_0 = \exp(p', tv')$ be a point on the boundary that is closest to y. Consider the path $\tilde{\gamma}$ obtained by concatenating γ_v and $\exp(p', tv') = \sigma$. Now $d(x, p') \leq l(\tilde{\gamma}) = T + \epsilon'$. Just as in the above claim, $l - T = d(p', y) = d(p'_0, y) + \epsilon$. Hence, by the triangle inequality, $l = d(x, y) \leq d(x, p'_0) + d(p'_0, y) = d(x, p'_0) + l - T - \epsilon'$ and hence

$d(x, p_0') \geq T + \epsilon'$. Thus $d(x, p_0') = T + \epsilon'$ and so $\tilde{\gamma}$ is a unit-speed minimal geodesic connecting x and p_0'. Thus it is smooth and by uniqueness, coincides with $\gamma_v(t)$. This also shows that $d(p_0', y) = l - (T + \epsilon')$ (where ϵ' is arbitrarily small) and hence contradicts the maximality of T.

□

Here are a few related theorems (some of which need Hopf-Rinow): Ambrose's theorem: Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds, and $\phi : \tilde{M} \to M$ be a local isometry. If $\tilde{M}, \tilde{g}$ is complete, then

1. ϕ is a covering map (and hence $\tilde{M}$ is a Riemannian cover).

2. (M, g) is complete.

The main point is this lemma:

Lemma 3.1. *Let $(\tilde{M}, \tilde{g})$ be complete. If $\phi : \tilde{M} \to M$ is a surjective local diffeo, and there exists a $c > 0$ such that $\|\phi_* v\| \geq c\|v\|$, then ϕ is a covering map.*