1 Recap

1. Proved theorem on existence of geodesically convex neighbourhoods.
2. Hopf-Rinow theorem.
3. Stated Ambrose's theorem (and a lemma).

2 Hopf-Rinow theorem and its consequences

Here are a few related theorems (some of which need Hopf-Rinow).

1. Ambrose's theorem: Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds, and $\phi : \tilde{M} \to M$ be a local isometry. If $\tilde{M}, \tilde{g}$ is complete, then

   (a) ϕ is a covering map (and hence $\tilde{M}$ is a Riemannian cover).

   (b) (M, g) is complete.

   Proof. The main point is this lemma:

   Lemma 2.1. *Let $(\tilde{M}, \tilde{g})$ be complete. If $\phi : \tilde{M} \to M$ is a surjective local diffeo, and there exists a $c > 0$ such that $\|\phi_* v\| \geq c\|v\|$, then ϕ is a covering map.*

   Proof. We first prove a path-lifting property for C^1 paths. Note that any lift is unique (given the starting point) by the local diffeo property (why?) Let γ be a C^1 path in M starting at p. Let $\tilde{p}$ be a point in the pre-image of p. Let S be the set of all t such that there is a C^1 lift starting at $\tilde{p}$ for time t. Note that S is non-empty (obviously), and it is open (by the local diffeo property). We need to prove that S is closed (and hence we will be done). Let $T = \sup S$ and $t_k \to T$ (with t_k increasing to T). The points $\tilde{\gamma}(t_k)$ is bounded, and hence by Hopf-Rinow, contained in a compact set: Indeed, $L(\gamma) \geq \int_0^{t_k} \|\gamma'\| \geq cd(\tilde{\gamma}(t_k), p)$.
   Thus $\tilde{\gamma}(t_k) \to \tilde{q}$ after passing to a subsequence. Choosing a neighbourhood around $\tilde{q}$ which is diffeo to one in M, we see that $\phi(\tilde{q}) = q$ is such that $\gamma(t_k) \to q$ and by continuity, $q = \gamma(T)$. Now we can extend the lift using the local diffeomorphism property. It turns out that this path-lifting property is enough to be a cover. This is a point set topology fact. (Roughly speaking, one can take a small enough coordinate ball around a point p such that it is diffeomorphic to some neighbourhood of some pre-image point, and consider it as an embedding. One can lift this embedding upstairs by lifting each radial straight line to get a bunch of open sets in the preimage. It turns out that they do not intersect with each other because of the unique lifting property and so on.) □

   Note that because ϕ is a local isometry, lengths of lifts are preserved. Moreover, ϕ takes geodesics to geodesics. (But not necessarily minimal ones to minimal ones.) There are a few steps:

   (a) Geodesic lifting: This is easy.

(b) Surjectivity: We argue by contradiction. ϕ is clearly an open map. Note that the image cannot be closed if it is not surjective (by connectedness of M). Let $p \in \phi(M)^c$ and let $B(p, \epsilon)$ be a normal geodesic ball. It contains a point $q = \phi(\tilde{q}) = \exp(tv)$. Now $\exp(tv)$ is a geodesic connecting p and q and hence has a unique lift upstairs. Thus p is in the image of ϕ. A contradiction. This means the map is a covering map.

(c) Completeness: Given a geodesic, and a point on it, we have a geodesic lift starting at a pre-image of that point that exists for all time. Taking ϕ, we conclude that the original geodesic exists for all time.

As a corollary, if (M, g) is complete, and $\exp_p$ has no critical points, then $\exp_p : T_pM \to M$ is a universal covering map. (We shall see later that if $sec \leq 0$, then $\exp_p$ has no critical points.)
Proof: Indeed, if $\exp_p$ has no critical points, then it is a local diffeomorphism. By Hopf-Rinow, it is surjective. Endowing T_pM with the pullback metric, we see that the metric is complete (indeed, from Hopf-Rinow, it is enough for $\exp(p, .)$ to exist on T_pM for one single p, and for that p, the geodesics of the pullback metric are $t(v^1, \ldots)$ which obviously exist for all of time to come) and hence $\exp_p$ is a covering map. □

2. Isometries: Let (M, g) and (N, h) be Riemannian manifolds. Suppose $\phi(p) = \psi(p)$, $\phi_*(p) = \psi_*(p)$, and that ϕ, ψ be isometries from M to N. Then $\phi \equiv \psi$.
Proof: Let S be the set of all points where ϕ, ψ coincide to first order. S is clearly non-empty and closed. It is also open: Let $q \in S$ and U be a normal neighbourhood. Then any $x \in U$ is $x = \exp_q(v)$ for some v. Using uniqueness of geodesics, we are done.

3. Geodesics in homotopy classes: Suppose (M, g) is a complete Riemannian manifold (always connected by assumption), and $p, q \in M$. Every path-homotopy class of admissible paths from p to q contains a minimising geodesic.
Proof: The universal cover $(\tilde{M}, \pi^* g)$ is complete (because geodesics can be lifted). All paths in a path-homotopy class can be lifted uniquely (given $\tilde{p}$ over p) and their endpoints are the same $\tilde{q}$ over q (standard fact). Since the universal cover is complete, there is a length minimising geodesic joining $\tilde{p}$ and $\tilde{q}$ (which is homotopic to all the paths joining p and q because of simple-connectedness). Since the covering map takes geodesics to geodesics, we have a geodesic joining p and q. Now if σ is any admissible curve joining p and q, then since $L(\tilde{\sigma}) = L(\sigma) \geq L(\tilde{\gamma}) = L(\gamma)$, we are done.
Remark: One can prove that the path-homotopy classes of admissible paths (via admissible homotopies, i.e., H(s,t) such that H is smooth on rectangles, it is continuous, and for each s, $H(s, t)$ is an admissible path) is the same as the path-homotopy classes of continuous paths. Indeed, given a continuous path, we claim that it is path homotopic to an admissible one: Cover it with coordinate charts. In each chart, approximate the continuous path by piecewise linear ones. If the approximation is good enough, then the straight line homotopy will be within the coordinate chart. We can also similarly prove that if two admissible paths are homotopic by a continuous map, then they are so by an admissible one.