1 Recap

1. Ambrose's theorem.
2. Corollary that if $\exp_p$ does not have critical points on complete manifold, then it is a covering map. (One crucial step is that thanks to Hopf Rinow, to prove completeness it is enough to prove that $\exp_p$ is defined on T_pM for one single point p!)
3. Theorem on isometries coinciding.
4. Theorem on path homotopy classes.

2 Second variation formula and Jacobi fields

Just as there is a second derivative test in one-variable calculus, one can attempt to study local and possibly global behaviour of geodesics using a second variation. Let $\gamma : [a, b] \to M$ be an admissible path. An admissible two-parameter variation $\Gamma : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times [a, b] \to M$ is a continuous map such that $\Gamma(0, 0, t) = \gamma(t)$ and on $(-\epsilon, \epsilon)^2 \times [a_{k-1}, a_k]$ it is smooth. We see that Γ is smooth in the variation variables, and there are two variation vector fields. Here is the second-variation formula.
Let γ be a geodesic, and Γ be an admissible proper two-parameter variation. Then

$$\begin{aligned}\partial^2_{s_1,s_2}|_{s_1=s_2=0}E &= \int_a^b \left(\langle D_t W_1, D_t W_2\rangle - Rm(W_1, \gamma', \gamma', W_2)\right) dt \\ &= -\sum_{k=1}^{m-1} \langle \Delta_k D_t W_1, W_2\rangle - \int_a^b \langle \frac{D^2 W_1}{dt^2} + R(W_1, \gamma')\gamma', W\rangle dt \end{aligned} \tag{1}$$

Proof.

$$\begin{aligned}\partial_{s_2} E &= \int_a^b \langle D_{s_2}\partial_t \Gamma, \partial_t \Gamma\rangle dt = \int_a^b \langle D_t \partial_{s_2}\Gamma, \partial_t\Gamma\rangle dt \\ \Rightarrow \partial^2_{s_1 s_2}|_{s_1=s_2=0}E &= \int_a^b (\langle D_{s_1} D_t \partial_{s_2}\Gamma, \gamma'\rangle + \langle D_t W_2, D_t W_1\rangle) dt \\ &= \int_a^b (\langle D_t D_{s_1}\partial_{s_2}\Gamma + R(W_1, \gamma')W_2, \gamma'\rangle + \langle D_t W_1, D_t W_2\rangle) dt \\ &= \int_a^b (\partial_t \langle D_{s_1}\partial_{s_2}\Gamma, \gamma'\rangle - \langle D_{s_1}\partial_{s_2}\Gamma, D_t\gamma'\rangle - \langle R(W_1, \gamma')\gamma', W_2\rangle + \langle D_t W_1, D_t W_2\rangle) dt \\ &= \langle D_{s_1}\partial_{s_2}|_{s_1=s_2=0}\Gamma(b-) - \langle D_{s_1}\partial_{s_2}|_{s_1=s_2=0}\Gamma(a+) \\ &+ \sum \Delta_k \langle D_{s_1}\partial_{s_2}|_{s_1=s_2=0}\Gamma, \gamma'\rangle + \int_a^b (-\langle R(W_1, \gamma')\gamma', W_2\rangle + \langle D_t W_1, D_t W_2\rangle) dt. \end{aligned} \tag{2}$$

Since the variation is proper, the first term vanishes. γ' is smooth at $s_1 = s_2 = 0$. Moreover, just as earlier, the s-derivatives of the variation fields are continuous (along with the variation fields themselves). Thus the second term vanishes too.
Now integration-by-parts completes the proof. □

As a corollary, if γ is of unit-speed, then

$$\partial^2_{s_1,s_2}|_{s_1=s_2=0}L = \int_a^b \left(\langle D_tW_1, D_tW_2\rangle - \langle D_tW_1,\gamma'\rangle\langle D_tW_2,\gamma'\rangle - Rm(W_1,\gamma',\gamma',W_2)\right)dt$$
$$= \int_a^b \left(\langle D_tW_1^{\perp}, D_tW_2^{\perp}\rangle - Riem(W_1^{\perp},\gamma',\gamma',W_2^{\perp})\right)dt. \tag{3}$$

Now we define the so-called Index form (the "Hessian" of the energy) at a geodesic γ as $I_\gamma(W_1,W_2) := \int_a^b (\langle W_1', W_2'\rangle - Rm(W_1,\gamma',\gamma',W_2))\,dt$ for two admissible fields and it is also equal to $\langle W_1', W_2\rangle|_a^b - \sum\langle \Delta_k W_1', W_2\rangle - \int_a^b \langle W_1'' + R(W_1,\gamma')\gamma', W_2\rangle dt$. Suppose we consider fields that vanish at the endpoints (the so-called "tangent space" at γ).
Then,

Lemma 2.1. *Let γ be a minimal geodesic. Then I_γ is positive-semidefinite on the "tangent space to γ".*

Proof. Take a variation with W as the variation field (like $\Gamma(s,t) = \exp(\gamma(t), sW(t))$). By Cauchy-Schwarz, E attains its minimum too and hence the second derivative test and the second variation formula prove the result. ☐

Suppose we take a variation by geodesics. What can we say about the variation vector field of such a variation? (Why do we care? $J(t) = ((\exp_p)_*)_{tv}(tw)$ is the variation field of $\Gamma(s,t) = \exp_p(t(v+sw))$ and we want to know about the critical points of this map.) One can easily compute and see (how?) that

$$J'' + R(J,\gamma')\gamma' = 0.$$

Vector fields along γ satisfying this equation are called Jacobi fields. (Note that the variation vector field above satisfies $J(0) = 0$ and $J'(0) = w$.)
Here is a proof that every Jacobi field gives rise to a variation of geodesics with this Jacobi field as the variation field: Let $\sigma : (-\epsilon,\epsilon) \to M$ be any smooth path such that $\sigma(0) = \gamma(0)$ and $\sigma'(0) = J(0)$. Then let $X(s), W(s)$ be parallel transports of $\gamma'(0), J'(0)$ along σ. Define $\Gamma(s,t) = \exp(\sigma(s), t(X(s) + sW(s)))$. Now for every fixed s, $\Gamma(s,t)$ is a geodesic. At $s = 0$, $\Gamma(0,t) = \exp(\gamma(0), t\gamma'(0)) = \gamma(t)$. Thus this is a variation through geodesics and hence the variation field V is a Jacobi field with $J(0) = \sigma'(0)$ and $V'(0) = D_t\partial_s\Gamma = D_s\partial_t\Gamma = D_s(X(s) + sW(s)) = W(0) = J'(0)$. Since the Jacobi field satisfies a second-order ODE, by uniqueness, $V = J$ throughout.
As a corollary, for a radial geodesic in geodesic normal coordinates, if the unique Jacobi field with $J(0) = 0$ and $J'(0) = w$ is $J(t) = tw$.
Given $J(a), J'(a)$ there is a unique Jacobi field with these starting values (and hence the space of Jacobi fields is $2n$-dimensional.) Indeed, this follows from ODE theory.