1 Logistics

Email: vamsipingali@iisc.ac.in. Course webpage: `http://math.iisc.ac.in/vamsipingali/ma333RiemGeom2024autumn/ma333.html`. HW : 20% (Roughly once in two weeks. Copying (from each other or the internet) is strictly not allowed.), Midterm - 30%, and Final (or project presentations - to be decided later) - 50%.

2 What is this course about and why should you care?

Riemannian geometry studies distances (and angles) on "curved" objects (like the Earth). In $\mathbb{R}^2$, we are familiar with Euclidean geometry, with its similarity, congruence, sum-of-interior-angles-of-a-triangle, circles, parallel lines, etc. Euclid formalised the axioms of this plane geometry using 5 axioms (actually, these axioms are not as rigorous as Euclid thought. The correct formulation awaited several centuries - the advent of Hilbert). The 5th axiom (the parallel postulate) stated that through every point not lying on a given line, there exists a unique line that does not intersect the given line. This postulate was thought to be a consequence of the other postulates but hyperbolic geometry (due to Bolyai and Lobachevsky) was a counterexample (wherein one removes uniqueness). If one allows other axioms to be violated, then spherical geometry is another counterexample. This led to interest in non-Euclidean geometries. These geometries are strange in that the sum of angles of triangle can be larger or smaller than 180 degrees. Moreover, Euclidean, hyerbolic, and spherical geometries are rather symmetric, in that, using an isometry, one can bring any point to any other point. More general distance functions (even in $\mathbb{R}^2$) do not satisfy this property of symmetry.

Riemann introduced the notion of an infinitesimal distance function (the Riemannian metric) on general objects, namely, manifolds. He developed (earlier notions due to Gauss) the concept of curvature on such objects. But other than for historical (accidental) reasons, why bother studying Riemannian geometry (or its variants like Lorentzian geometry)?

1. The cartographical question of whether we can draw a map of any part of the earth to scale. (The answer is - we cannot, thanks to Gauss's Theorema Egregium.)
2. General relativity due to Einstein requires Lorentzian metrics.
3. Image processing: How can a computer recognise say, known criminals performing crimes? (Congruence)
4. Topology (Poincaré conjecture proved using Riemannian geometry), Algebraic geometry (the Bogomolov-Miyaoka-Yau inequality), and number theory (symmetric spaces and Harish Chandra).

In ordinary Euclidean geometry, we have congruence results (like the SAS property for triangles) and local-to-global results (like the sum of angles being 180 degrees or more generally, the sum of exterior angles of a polygon being 2π radians). We aim at similar results in Riemannian geometry. Congruence is replaced by "Are these two Riemannian manifolds isometric?" and local-to-global results are more complicated (like the Gauss-Bonnet-Theorem and a vast reaching generalisation - the Atiyah-Singer index theorem).

3 Plane curves, curvature, and a local-to-global result

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a piecewise smooth path. We say that it is closed if $\gamma(a) = \gamma(b)$ and simple if γ is $1 - 1$ except possibly at a, b. It is regular if at all smooth points, $v(t) = \gamma'(t) \neq 0$. At such points, we can define the unit tangent vector $T(t) = \frac{v(t)}{\|v(t)\|}$. We can also reparametrise each smooth piece using the arc-length $s = \displaystyle\int_{t_1}^{t_2} \|v(t)\| dt$ to get a path whose velocity is 1. It is easy to see that any arc-length parametrised path has $\langle a(s), v(s) \rangle = 0$ where $a(s) = v'(s)$ is the acceleration. Using the centripetal acceleration as motivation, we define the curvature of a smooth arc-length parametrised path as $\kappa(s) = \|a(s)\|$. Clearly, a line has zero curvature and a circle has constant curvature (approaching zero as the radius gets large). We now specialise to planar paths.

A curved polygon is a piecewise smooth arc-length parametrised simple closed path, such that for each of the vertices, $v(a_i-) \neq -v(a_i+)$ and $v(0+) = v(l-)$. We now introduce the notion of signed curvature. Consider the unit normal $N = (-T_2(t), T_1(t))$. Then $k_g(s) = \langle a(s), N(s) \rangle$. Thus $a(s) = k_g(s)N(s)$. This curvature is positive if the curve bends towards the normal. With this convention, the signed curvature of a circle is positive (if we move in the anticlockwise direction).

We now want to prove a generalisation of the sum-angle property for curved planar polygons. To this end, we need to be able to define the "angle" of a curve:
Lemma (Tangent angle lemma): For any unit-speed smooth path $\gamma : I \to \mathbb{R}^2$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that $\gamma'(s) = (\cos(\theta)(s), \sin(\theta)(s))$. Then $k_g(s) = \theta'(s)$.

Proof. Note that γ' is a smooth map to S^1. The unique lifting lemma states that there is a unique smooth lift to the universal covering space (with the map $\exp(t) = e^{it}$) if we fix $\theta(t_0) = \theta_0$. Actually, the lemma states that there is a unique continuous lift. However, since the covering map is smooth, it is easy to see (locally using the evenly covered neighbourhoods) that this lift is smooth. A calculation now shows what we need. □

We can now state Hopf's rotation theorem: $\displaystyle\int_0^l k_g(s)ds + \sum_k \epsilon_k = 2\pi$ where the vertex defect/rotation $\epsilon_k = \theta(a_k+) - \theta(a_k-)$.

Proof. First assume that γ is smooth. Then $\displaystyle\int_0^l k_g(s)ds = \theta(l) - \theta(0)$. By translating and rotating (these operations are isometries and don't change anything), assume that $\gamma(0) = \gamma(l) = 0$ and that γ is in the upper half plane. We now compare $\theta(s)$ to the angle made with the x-axis, i.e., the secant-angle function $\phi(s_1, s_2)$ defined on the triangle $T = \{0 \leq s_1 \leq s_2 \leq l\}$ as the unique continuous lift with $\phi(0, 0) = 0 = \theta(0)$ of the continuous function

$$\psi(s_1, s_2) = \frac{\gamma(s_2) - \gamma(s_1)}{\|\gamma(s_1) - \gamma(s_2)\|} \; if \; s_1 < s_2, \; (s_1, s_2) \neq (0, l)$$
$$= v(s_1), \; if \; s_1 = s_2$$
$$= -v(0), \; (s_1, s_2) = (0, l). \tag{1}$$

Note that $v(s) = \psi(s, s)$ and thus $\phi(s, s) = \theta(s)$. Now since γ is in the upper half plane and $\phi(0, 0) = 0$, we see that $\sin(\phi(0, s)) \geq 0$ and hence $\phi(0, s) \in [0, \pi]$. In particular, $\phi(0, l) = \pi$. Likewise, $\psi(s, l)$ has negative y-coordinate and since $\phi(0, l) = \pi$, $\phi(l, l) = 2\pi$. Thus $\theta(l) - \theta(0) = 2\pi$.

If γ is not smooth, smoothly "round off" the corners (using arcs of hyperbolae) so that the tangent angle monotonically changes by the defect, and apply the first part of the proof. (Details are in Lee's book.)

The congruence theorem for plane curves is: Two unit-speed equal length plane curves are congruent by direction-preserving congruence iff their signed curvatures are equal. This theorem will be left as a HW exercise. □