1 Recap

1. Second variation formula for length and energy.
2. Index form is positive-semidefinite on proper variation fields for minimal geodesics.
3. Motivation of Jacobi fields from families of geodesics converging or diverging (and tidal forces). Jacobi equation. Existence, uniqueness, dimension.
4. Proof that Jacobi fields correspond to variation fields via geodesics.

2 Second variation formula and Jacobi fields

Clearly, $J_0(t) = \gamma'$ is a Jacobi field but it is a "trivial" Jacobi field. (It satisfies $J(a) = \gamma'(a), J'(a) = 0$.) So is $J_1 = t\gamma'(t)$. (It satisfies $J(a) = 0, J'(a) = \gamma'(a)$.)
We make a definition: A normal vector field along a curve is an admissible vector field that is perpendicular to the curve.
We claim that the space of Jacobi fields that is normal to a geodesic is a vector space of dimension $2n-2$ consisting precisely of those Jacobi fields that are perpendicular to J_0 and J_1 (with the L^2 inner product). Indeed, firstly, $f(t) = \langle J(t), \gamma'(t)\rangle$ is linear (because $f''(t) = 0$) and hence is zero identically iff $f(a) = f(b) = 0$ for $a \neq b$. Also it is zero iff $f(a) = f'(a) = 0$. The latter implies that indeed such normal Jacobi fields are in $1-1$ correspondence with $v, w \in T_{\gamma(0)}M$ such that $v, w \perp \gamma'(0)$. Thus the space of fields is $2n-2$ dimensional. Now any Jacobi field W that is perpendicular to J_0, J_1 is normal (why?) and hence the space of normal fields is precisely this space.
Note that v is a critical point of $\exp_p$ if there is a w such that $J(1) = 0$, i.e., there is a Jacobi field along the geodesic γ so that $J(0) = 0 = J(1)$. If a point p and q can be connected by a geodesic such that such a Jacobi field exists, then q is said to be conjugate to p. (Note that affine changes of time do not change the Jacobi equation and hence, if p is conjugate to q iff q is conjugate to p.)

3 Applications of Jacobi fields

One important point is that geodesics stop minimising after the first conjugate point. Roughly the idea is that at a conjugate point, there is a family of geodesics that almost end at the same point. So now we can create a " broken geodesic" with the same length and hence round off the corner to get something smaller. An infinitesimal version of this argument proves the following theorem.

Theorem 1. *Let $\gamma : [a, b] \to M$ be a unit-speed geodesic from p to q.*

1. *If there is a $t_0 \in (a, b)$ such that $\gamma(t_0)$ is conjugate to p along γ, then there is a proper normal variation field such that $I_\gamma(X, X) < 0$. Thus γ is not minimal.*
2. *If there is no interior conjugate point but $\gamma(a)$ and $\gamma(b)$ are conjugate, then for every proper variation, the curve $\Gamma(s, .)$ is strictly longer than γ for all sufficiently small s unless the variation field is Jacobi. If γ has no conjugate points, then for every proper variation, the curve $\Gamma(s, .)$ is strictly longer than γ for sufficiently small s.*

Proof. 1. Suppose $\gamma(c)$ is conjugate to p along γ. Then there is a nontrivial normal (why?) Jacobi field that vanishes at a and b. Now define a piecewise smooth normal field $V = J$ on $[a, c]$ and 0 otherwise. Using an orthonormal frame and a bump function, we can now construct a proper normal field W such that $W(c) = \Delta D_t V = -D_t J(c) \neq 0$ is the jump (why is it non-zero?). Let $X_\epsilon = V + \epsilon W$ for $\epsilon > 0$. Then $I(X_\epsilon, X_\epsilon) = I(V, V) + 2\epsilon I(V, W) + \epsilon^2 I(W, W)$. Now $I(V, V) = 0$ (because recall that $I(V, W) = -\int_a^b \langle D_t^2 V + R(V, \gamma')\gamma', W\rangle dt + \langle D_t V, W\rangle|_a^b - \sum\langle \Delta_i D_t V, W(a_i)\rangle$). Now $I(V, W) = -|W(b)|^2$ and hence for sufficiently small ϵ, $I(X_\epsilon, X_\epsilon) < 0$. Thus $E(\gamma_s) < E(\gamma)$ for a variation with variation field X_ϵ. By Cauchy-Schwarz, we see that the length is strictly larger.

2. The idea is to choose a nice frame along γ such that we can compute $I(V, V)$ easily. (After all, the aim is to prove that $I(V, V) \geq 0$ with equality iff V is Jacobi.) Now we can definitely parallel transport an onb but it does not help with the Riemann term in the expression for $I(V, V)$. The Jacobi equation on the other hand has a Riemann term and hence we shall use that instead. Assume that $a = 0$ and let $w_1, \ldots, w_n$ be an onb for $T_p M$ with $w_1 = \gamma'(0)$. Let J_α be the Jacobi field with $J_\alpha(0) = 0$ and $J'_\alpha(0) = w_\alpha$ (for $\alpha \geq 2$). Since there are no interior conjugate points, $J_\alpha(t)$ forms an basis for the orthocomplement of $\gamma'(t)$ in $T_{\gamma(t)}M$ and hence for a given variation field V, $V = v^\alpha J_\alpha$ where $v_\alpha : (0, b)$ are piecewise smooth. To be continued.....

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