1 Recap

1. Normal Jacobi field. Conjuate points.
2. Statement of the theorem that basically says geodesics stop minimising beyond the first conjugate point.

2 Applications of Jacobi fields

Theorem 1. *Let $\gamma : [a, b] \to M$ be a unit-speed geodesic from p to q.*

1. *If there is a $t_0 \in (a, b)$ such that $\gamma(t_0)$ is conjugate to p along γ, then there is a proper normal variation field such that $I_\gamma(X, X) < 0$. Thus γ is not minimal.*
2. *If there is no interior conjugate point but $\gamma(a)$ and $\gamma(b)$ are conjugate, then for every proper normal variation, the curve $\Gamma(s, .)$ is strictly longer than γ for all sufficiently small s unless the variation field is Jacobi. If γ has no conjugate points, then for every proper normal variation, the curve $\Gamma(s, .)$ is strictly longer than γ for sufficiently small s.*

Proof. 1. Last time.

2. Assume that $a = 0$ and let $w_1, \ldots, w_n$ be an onb for T_pM with $w_1 = \gamma'(0)$. Let J_α be the Jacobi field with $J_\alpha(0) = 0$ and $J'_\alpha(0) = w_\alpha$ (for $\alpha \geq 2$). Since there are no interior conjugate points, $J_\alpha(t)$ forms an basis for the orthocomplement of $\gamma'(t)$ in $T_{\gamma(t)}M$ and hence for a given normal variation field V, $V = v^\alpha J_\alpha$ where $v_\alpha : (0, b)$ are piecewise smooth. In fact, v_α have a smooth extension to $[0, b)$: Indeed, in geodesic normal coordinates, $J_\alpha(t) = t\partial_\alpha$ (indeed, one can see that this field satisfies the Jacobi ODE (why?)). By Taylor's theorem the components of $V(t)/t$ extend smoothly. Likewise extension happens at $t = b$ as well.
Let $X = v^\alpha J'_\alpha$ and $Y = (v^\alpha)' J_\alpha$. Then $I(V, V) = \int_0^b (|V'|^2 - Riem(V, \gamma', \gamma', V))\, dt$. Now note that the Riemann tensor term (which is the problematic one) is $v^\alpha \langle J''_\alpha, V\rangle$ and hence

$$I(V, V) = \int_0^b \left(|V'|^2 + \langle X', V\rangle - (v^\alpha)'\langle J'_\alpha, V\rangle\right) dt$$
$$= \int_0^b \left(\langle X, V'\rangle + \langle Y, V'\rangle + \langle X, V\rangle' - \langle X, V'\rangle - (v^\alpha)' v^\beta \langle J'_\alpha, J_\beta\rangle\right) dt$$
$$= \int_0^b \left(|Y|^2 + \langle X, V\rangle' + (v^\alpha)' v^\beta \langle J_\alpha, (J_\beta)'\rangle - (v^\alpha)' v^\beta \langle (J_\alpha)', J_\beta\rangle\right) dt. \quad (1)$$

Now

$$f(t) = \langle J'_1, J_2\rangle - \langle J_1, J'_2\rangle$$
$$\Rightarrow f' = \langle -R(J_1, \gamma')\gamma', J_2\rangle + \langle J_1, R(J_2, \gamma')\gamma'\rangle = 0. \quad (2)$$

Hence $I(V, V) \geq \sum \langle X, V\rangle_{a_i}^{a_{i+1}} = 0$ with equality implying that $Y = 0$, i.e., v_α is a constant and hence V is a Jacobi field. Since it is a proper variation, p and $\gamma(b)$ are conjugate to each other. If there are no conjugate points, then $Y \neq 0$ and hence $I(V, V) > 0$.

In other words, a geodesic stops being minimising beyond its first conjugate point. Motivated by this property, let's define: Let (M, g) be a (connected as always) complete Riemannian manifold, $p \in M, v \in T_pM$. Define the cut time $t_{cut}(p, v)$ of (p, v) as the sup of all $b > 0$ such that γ_v restricted to $[0, b]$ is minimising. Clearly $t_{cut}(p, v) > 0$ but it can be $< \infty$. Moreover, $t_{cut}(p, v)$ is uniformly positive for a neighbourhood of $(p, v) \in TM$. If it is finite, the cut point of p along γ_v is $\gamma_v(t_{cut})$. The collection of cut points of p is called the cut locus of p $Cut(p)$. Note that for a sphere with the round metric, the cut locus is a single point. For $\mathbb{R}^n$ with the Euclidean metric, there is no cut locus. Here is an important property of cut times.

Theorem 2. *Let (M, g) be complete, $(p, v) \in TM$ and $|v| = 1$. Then*

1. *If $0 < b < t_{cut}(p, v)$, then γ_v has no conjugate points on $[0, b]$ and is the unique unit-speed minimiser between its endpoints.*

2. *If $t_{cut}(p, v) < \infty$ and $\gamma(t_{cut})$ is not conjugate to p, then γ_v is minimising on $[0, t_{cut}]$ and there is at least one more minimsing geodesic between p and $\gamma_v(t_{cut})$.*

Proof. 1. If there is an interior conjugate point, then γ_v would have stopped minimising. γ_v is minimising on $[0, b]$ (why?) It is also unique (why?)

2. Suppose $c = t_{cut} < \infty$. Then γ_v is minimising on $[0, b_N = c - \frac{1}{N}]$. By continuity, $d(p, \gamma_v(c)) = \lim d(p, \gamma_v(b_N)) = \lim b_N = c$. Thus γ_v is minimising on $[0, c]$ (and hence $\gamma_v(c)$ cannot be a conjugate point to p) and not beyond. Consider $c_n = c + \frac{1}{n}$. There is a sequence of unit speed minimal geodesics $\gamma_{v_n}(t)$ connecting p and $\gamma_v(a_n)$. By compactness, there is a subsequence (that we continue to denote as v_n) such that $v_n \to w$ and by passing to a further subsequence, that $a_n = l(\gamma_{v_n})$ converges to l. Now $c = d(p, \gamma_v(c)) = \lim d(p, \gamma_{v_n}(a_n)) = \lim a_n = l$. Thus γ_w is also a minimising geodesic. We need to show that it is different from γ_v. Since $\gamma_v(c)$ is not conjugate, there is a neighbourhood V around this point where $\exp$ is injective. Now $\exp_p(a_n v_n) = \exp(c_n v)$ for some c_n but since $a_n < c_n$, $a_n v_n$ is not in V for large n. Hence the limit cw is also not in V for large n. Thus $w \neq v$.

□

We have the following important corollary.

Theorem 3. *Let (M, g) be complete and $B(0, r)$ be a ball in T_pM. Then if $\exp_p : B(0, r) \to M$ is injective, it is a diffeomorphism to its image.*

Proof. Obviously we only need to check whether the exponential map is an immersion. Suppose it is not at some point v. Then $(\exp_p)_*$ at v has a kernel, i.e., there is a conjugate point along $\exp_p(tv)$ at $t = 1$. Take the first conjugate point along this geodesic. Let's say it occurs at $t_0 \leq 1$. By the theorem above, $\exp_p(tv)$ stops being minimising starting from t_0. By completeness, there is one more geodesic $\exp_p(tw)$ that has shorter length. But this contradicts the injectivity of the exponential map. □

Motivated by this theorem, we define the injectivity radius of a point p as the sup over all r such that $\exp_p : B(0, r) \to M$ is a diffeomorphism. By the above theorem, it is precisely the set of points where the map is injective. The injectivity radius of (M, g) is the inf of all injectivity radii of points. Note that $inj(M, g)$ can be zero (example?) only for non-compact manifolds (why?) $\square$