1 Recap

1. Geodesics stop minimising beyond the first conjugate point.
2. Cut time, and injectivity radius.

2 Applications of Jacobi fields

We now proceed to calculate Jacobi fields on space-forms. The point is that Jacobi fields basically measure the deviation of geodesics. So we expect that if the curvature is largely positive, the geodesics will come close and diverge otherwise. To do this comparison, we need explicit formulae for the Jacobi fields of spaces of constant curvature. Define the functions

$$\begin{aligned} sn_K(t) &= \sqrt{K}\sin(\sqrt{K}t),\ K>0 \\ &= t,\ K=0 \\ &= \sqrt{-K}\sinh(\sqrt{-K}t),\ K<0. \end{aligned} \tag{1}$$

This function satisfies $sn_K'' + Ks = 0$ and $s(0) = 0, s'(0) = 1$.
Assume that the sectional curvature is a constant K and that $e_1 = \gamma', \ldots, e_n$ is a parallel collection of orthonormal vector fields on a unit-speed smooth geodesic γ.

$$\begin{aligned} \langle J, e_i\rangle'' = \langle J'', e_i\rangle &= -Riem(J, e_1, e_1, e_i) = -J^k Riem(e_k, e_1, e_1, e_i) \\ &= \frac{1}{2}J^k(R_{k11k} + R_{i11i} - Riem(e_k+e_i, e_1, e_1, e_k+e_i). \end{aligned} \tag{2}$$

If $k \neq i$, the right-hand-side is 0. If $k = i = 1$, it is 0. Otherwise, $\langle J, e_k\rangle'' = -J^k K$ if $k \neq 1$. Thus, if $J(0) = 0$, then $J = sn_K(t)E(t)$ for some normal parallel vector field $E(t)$.
As a corollary, we claim that if $sec \equiv K$, then in geodesic normal coordinates, $g = dr^2 + sn_K^2(r)g_{S^{n-1}}$. In particular, the geodesic normal coordinates give an isometry between this metric and the metric with constant sectional curvature. (The Killing-Hopf theorem gives a global version of this result.)
Proof: By Gauss, $g = dr^2 + h_r$. The Euclidean metric is $g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$ and coincides with g at p. Suppose v is a tangent vector (that is tangent to S_r) at $q = \exp_p(rv)$, then $|v|^2 = h_r$. Now we evaluate this number in another way, using Jacobi fields. Basically, we know the explicit formula for a Jacobi field with $J(0) = 0$ and $J'(0) = \frac{v}{r}$ along a radial geodesic. Indeed, it is $J = t\frac{v}{r}$. Note that J is perpendicular to γ at p, q and hence $J = sn_K(t)E(t)$ where $J'(0) = E(0) = \frac{v}{r}$ and $|E(t)| = |E(0)|$. Now, $|v|_q^2 = |J(r)|^2 = sn_K^2(r)|E(0)|^2 = sn_K^2\frac{|v|_p^2}{r^2} = sn_K^2 g_{S^{n-1}}(v,v)$ and hence we are done. $\square$

3 Comparison geometry

Here is a bunch of theorems that fall under the theme of “comparison geometry".

3.1 Bonnet-Myers

Let (M, g) be a complete manifold with $Ricc(X, X) \geq (n-1)Kg(X, X)$ for a constant $K > 0$. Then M is compact and $diam(M) \leq \frac{\pi}{\sqrt{K}}$.
Proof: The idea is that if the Ricci is bounded below, then since $Ricci$ is a sum of $(n-1)$ sectional curvatures, (indeed, $Ricc_{ii} = \sum_l R_{liil}$) we expect that some sort of a "Laplacian" (sum of Index forms) of the energy/length functional is bounded below at least for large geodesics and hence large geodesics cannot be length minimising. The sum of Index forms must each be evaluated on the Jacobi fields of a model space because we expect that the geodesics in M converge faster than the ones in the model space.
Indeed, suppose $l = d(p, q) > \frac{\pi}{\sqrt{K}}$ and $\gamma : [0, 1] \to M$ is a minimal geodesic. Let e_i be parallel orthonormal fields along γ such that $e_1 = \frac{\gamma'}{l}$. Then define $V_j = \sin(\pi t)e_j$. Then $V_j(1) = V_j(0) = 0$ and $\sum_{j\geq 2} I(V_j, V_j) < 0$ and hence for some j, $I(V_j, V_j) < 0$ and we have a contradiction. $\square$
It is crucial that one has a uniform lower bound. Indeed, take a paraboloid $z = x^2 + y^2$. It has $K > 0$ but is noncompact.
It turns out that if equality holds in the Ricci lower bound above, then (M, g) is isometric to a sphere with a round metric (Cheng's rigidity theorem).
Lastly, notice that the universal cover $(\tilde{M}, \pi^* g)$ is complete and continues to satisfy the hypotheses and is hence compact. This means that the fundamental group is finite! (In particular, it turns out that $dimH^1(M) = 0$ assuming some results of algebraic topology.)