1 Recap

1. Formula for the Jacobi fields of model spaces and an application to find the metric in geodesic normal coordinates.
2. Bonnet-Myers and a corollary.

2 Comparison geometry

2.1 Cartan-Hadamard

Let (M, g) be a complete manifold with $sec \leq 0$. Then the universal cover of M is diffeomorphic to $\mathbb{R}^n$.
Proof:We simply need to prove that $\exp_p$ does not have critical points for some fixed p. If indeed it did at q, then consider the minimal geodesic $\gamma_v(t)$ joining p and q with a Jacobi field J that vanishes at p and q. Now

$$\begin{aligned} J'' + R(J, \gamma')\gamma' = 0 \\ \Rightarrow \langle J'', J\rangle + Riem(J, \gamma', \gamma', J) = 0 \\ \Rightarrow \langle J'', J\rangle \geq 0 \; Rightarrow \langle J', J\rangle' \geq 0. \end{aligned} \tag{1}$$

with equality iff $J' = 0$. Thus using the endpoints we conclude that $J' \equiv 0$ and hence $J \equiv 0$. A contradiction □

2.2 Killing-Hopf

Theorem 1. *Let (M, g) be simply connected with constant sectional curvature K. Then (M, g) is isometric to (M_K, g_K) where M_K is the space form with sectional curvature K.*

Proof. When $K \leq 0$, we already know by Cartan-Hadamard's proof that the exponential map is a diffeomorphism. Now take any point $\tilde{p}$ in the space form and $p \in M$. Consider the map $F = \exp_p \circ \exp_{\tilde{p}}^{-1}$ (after choosing orthonormal bases for $T_{\tilde{p}}M_K$ and T_pM and identifying them with each other. The map F is a diffeomorphism from M_K to M. Since the metrics agree in normal coordinates, it is also an isometry.
By rescaling, assume that $K = 1$. (By Bonnet-Myers, we see that the diameter is $\leq \pi$.) We need to know about the critical points of the exponential map, i.e., conjugate points. To this end, we prove the following comparison theorem.

Theorem 2. *Let (M, g) be a complete manifold such that $sec \leq k$, for some $k > 0$. Let $\gamma : [0, l] \to M$ be unit-speed geodesics such that $\gamma(0)$ and $\gamma(l)$ are conjugate. Then $l \geq \pi$.*

Proof. Let $K = 1$ by rescaling. The idea is to compare this situation with that of a sphere (wherein we know the Jacobi fields). Let $p = \gamma(0)$, and $v = \gamma'(0)$. Assume without loss of generality that $\gamma(l)$ is the first conjugate point. Then there is a non-trivial normal

Jacobi field J such that $J(0) = J(l) = 0$ and $J(t) \neq 0$ on $(0, l)$. Normalise the Jacobi field so that $|J'(0)| = 1$. Let $u(t) = |J(t)|$. Note that u is smooth on $(0, l)$. It satisfies,

$$\begin{aligned} u' &= \frac{\langle J', J \rangle}{|J(t)|} \\ \Rightarrow u'' &\geq \frac{\langle J'', J \rangle}{u(t)} = -\frac{Riem(J, \gamma', \gamma', J)}{u(t)} \\ &\geq -u. \end{aligned} \tag{2}$$

Suppose $l < \pi$. Let $v(t) = \sin(t)$. Then $v'' + v = 0$. Now $(u'v - uv')' \geq 0$ and hence $f(t) = \frac{u(t)}{v(t)}$ satisfies $f' \geq 0$ on $(0, l)$. Thus $f(l) = 0 \geq f(0+)$. We claim that $f(0+) = 1$ and hence there is a contradiction. Indeed, in geodesic normal coordinates, $J(t) = tJ'(0)$. Thus $u(t) = t + O(t^3)$ and hence $f(0+) = 1$. □

As a consequence, $\exp_p$ is a local diffeomorphism on $B(0, \pi)$. Moreover, since $\exp_{\tilde{p}}$ is a diffeomorphism on $B(0, \pi)$ on the sphere (it only misses the opposite pole), the composition yields a local diffeomorphism ϕ from the sphere minus a pole to an open subset of M. Again using polar coordinates, we see that this map is actually a local isometry. Likewise, we have a local isometry ψ from the sphere minus the some point Q (such that $\psi(Q) = \phi(Q)$) which is neither of the poles to an open subset of M. By a previously proved lemma, $\phi = \psi$ wherever they are both defined. Hence we have a local isometry from S^n to M. By Ambrose, it is a cover and since S^n, M are simply connected, it is a global isometry. □

Here are some corollaries of the comparison principle in the proof above (Let (M, g) be complete and satisfy $sec \leq K$):

1. In geodesic normal coordinates, $g = dr^2 + h_r$ and $g_K = dr^2 + sn_K^(2)(r) g_{S^{n-1}}$. Then $h_r \geq sn_K^2(r) g_{S^{n-1}}$. In particular, $g(w, w) \geq g_K(w, w)$.
   As usual, suppose q is a point and $\gamma(t)$ is the radial geodesic connecting p and q. Then $J = t\frac{w}{r}$ where w is tangent to a sphere of radius r. By the comparison principle, $|J(r)|^2 = |w|_q^2 \geq \frac{sn_K^2}{r^2} |w|_{Euc}^2 = sn_K^2 |w|_{S^{n-1}}^2$.

2. If $K \leq 0$, $|((\exp_p)_*)_v(w)| \geq |w|$. In particular, for any curve σ in T_pM, we have, $L(\sigma) \leq L(\exp_p \sigma)$.
   Let $\gamma(t) = \exp_p(tv)$. Then if $J(0) = 0, J'(0) = w$, $J(t) = (\exp_p)_*)_{tv}(tw)$. Thus by the comparison theorem, $|J(1)| \geq |J'(0)| = |w|$. The second part follows easily.

3. Let $K \leq 0$ and ΔABC be a geodesic triangle. Then $A + B + C \leq \pi$ and $a^2 + b^2 - 2ab \cos C \leq c^2$.
   Let O be the origin in T_CM and $OA'B'$ a triangle in T_CM with $OA' = a, OB' = b$ and $O = C$. Now $\exp_C(A') = A$ and $\exp_C(B') = B$. By the previous corollary and the cosine rule, the second part follows.
   By the triangle inequality, there is a Euclidean triangle with sides a, b, c. Using the cosine rule and the second part, we are done.

2.3 Bishop-Gromov

Let (M, g) be complete and $Ric \geq (n-1)K$ where $K \in \mathbb{R}$. Define the volume ratio $V_K(r) = \frac{Vol(B_g(p,r))}{Vol(B_K(p_K,r))}$ where $p \in M$ and $p_K \in M_K$ (the choice of p_K does not matter). Then $V_K(r)$ is non-increasing on $(0, \infty)$. Thus $V_K(r) \leq V_K(0+) = 1$ and hence $Vol(B_g(p,r)) \leq Vol(B_K(p_K, r))$. (Also, if the diameter is finite, then $V_K(r) \geq V_K(R)$. This means the volumes of balls grow at least at a certain rate.)