1 Recap

1. Cartan-Hadamard
2. Killing-Hopf using ODE comparison
3. Triangle inequality comparison.
4. Statement of Bishop-Gromov.

2 Comparison geometry

2.1 Bishop-Gromov volume comparison and rigidity

Let (M, g) be complete and $Ric \geq (n-1)K$ where $K \in \mathbb{R}$. Define the volume ratio $V_K(r) = \frac{Vol(B_g(p,r))}{Vol(B_K(p_K,r))}$ where $p \in M$ and $p_K \in M_K$ (the choice of p_K does not matter). Then $V_K(r)$ is non-increasing on $(0, \infty)$ (actually it is non-increasing on $(0, inj_p)$ even if M is not complete). Thus $V_K(r) \leq V_K(0+) = 1$ and hence $Vol(B_g(p, r)) \leq Vol(B_K(p_K, r))$. (Also, if the diameter is finite, then $V_K(r) \geq V_K(R)$. This means the volumes of balls grow at least at a certain rate.) Moreover, if equality holds then $B(p, R)$ is isometric to $B_K(p_K, R)$ if $R < inj_p$. In general, the space has constant curvature. If $R > dia(M)$ and $K > 0$ then M is isometric to a sphere.
To prove such a result, we must differentiate the volume integral (and hence the volume form) in normal coordinates. Let $D_p = M - Cut_p$ and $\tilde{D}_p = T_pM - \exp^{-1}(Cut_p)$. On D_p we have geodesic normal coordinates (the exponential map is a diffeomorphism). Now if $q = \exp_p(v)$, then $\partial_i|_q = ((\exp_p)_*)_v(e_i)$. Let S_pM be the unit sphere in T_pM. The volume form is $vol_g = \sqrt{\det(g)}dx^1 \dots dx^n$ (where x^i are arranged to be an oriented chart). Now in the exponential polar coordinates, $vol_g = A(r, \theta)drd\sigma_{n-1}$ where $A(r, \theta) = r^{n-1}\det(((\exp_p)_*)_{r,\theta})$. Now as always, we want to write this expression using Jacobi fields so that we can use the Jacobi equation to differentiate. We claim that if $J_2, \dots, J_n$ (because $J_1 = \gamma'$) are normal Jacobi fields along $\gamma_\theta(t) = t\theta$ (where $\theta \in S_pM$) such that $J_2'(0), \dots$ are linearly independent and $J_i(0) = 0$, then

$$A(t, \theta) = \frac{|\gamma' \wedge J_2(t) \wedge \dots|_{\gamma_\theta(t)}}{|J_1'(0) \wedge \dots|_p}. \tag{1}$$

Suppose $J_i'(0) = w_i$ (with $J_1'(0) = w_1 = \gamma'(0)$), then J_i (for $i \neq 1$) is the variation field of $\Gamma_i(t, s) = \exp_p(t(\theta + sw_i))$ and $J_i(t) = ((\exp_p)_*)_{t\theta}(tw_i)$. That is, the matrix whose columns are J_i (including J_1) is $D\exp_p[\gamma' \; tW]$. Thus the determinant of this matrix is $\det(D\exp_p)t^{n-1}\det([\gamma' \; W])$ and hence we are done. $\square$
As a corollary, If (M_k, g_K) is a constant sec$=K$ simply connected complete manifold, then $A_K(r, \theta) = sn_K^{n-1}(r)$. (Indeed, the Jacobi's are $sn_k(t)e_k(t)$ where $e_k(t)$ are parallel transports of an orthonormal basis.)
Here is an interesting consequence: we can calculate the area of S^n as $\int_{S^{n-1}}\int_0^\pi \sin^{n-1}(t)dtd\sigma_{n-1} = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$ where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ when $s > 0$.
Now we differentiate the volume form.

Lemma 2.1. *If $Ricc \geq (n-1)Kg$, then for all $t < t_{cut}(\theta)$,*

$$\frac{d}{dt}\ln A(t,\theta) \leq \frac{d}{dt}\ln A_K(t,\theta) = (n-1)\frac{sn'_K}{sn_K}. \tag{2}$$

Thus $A(t,\theta) \leq A_K(t)$. Moreover, equality holds for some $t = R$ and all θ iff $B(p,R)$ is isometric to $B_K(R)$.

Proof. The first step is to realise that for any invertible matrix-valued path, $\frac{d}{dt}\ln A = tr(A^{-1}A')$. This equality can be proven for matrices with distinct eigenvalues by the implicit function theorem and diagonalisation. For general matrices, noting that the derivative at $t = 0$ is the same as for the path $A(t) = A(0) + tA'(0)$ and that both sides are real analytic in t as well as in the entries of A, it holds for all invertible matrices.
Choose $\gamma'(0), e_2, \ldots, e_n$ to be an oriented orthonormal frame for T_pM, parallel transport to get $\gamma', E_2, \ldots, E_n$ along γ. On an oriented manifold, parallel transport preserves orientation (why?) Thus $\gamma', E_2 \ldots$ is oriented. Fix $T < t_{cut}$. Let J_i be the Jacobi field with $J_i(0) = 0$, $J_i(T) = E_i(T)$. (The two-point problem has a solution provided there are no conjugate points.) Let $A(t)$ be the matrix taking $\gamma', E_2, \ldots$ to $\gamma', J_2, \ldots$. This matrix is non-singular on $(0, T]$ and hence has positive determinant. Moreover, $\det(A)^2 = \det(P)$ where $P_{kl} = \langle J_k, J_l\rangle$. Thus, $\frac{d}{dt}\ln A(t,\theta) = \frac{1}{2}tr(P^{-1}P')$ which at $t = T$ is $\sum_i \langle J'_i, J_i\rangle(T) = \sum_i I(J_i, J_i)$ which by the following index lemma is $\leq \sum_i I(X_i, X_i)$ where $X_i = \frac{sn_K(t)}{sn_K(T)}E_i(t)$:

Lemma 2.2. *Let $\gamma : [0, b] \to M$ be a unit-speed geodesic with no conjugate points. If J is a normal Jacobi field and X is any normal field along γ such that $X(0) = J(0) = 0$ and $X(b) = J(b)$, then $I_\gamma(J, J) \leq I_\gamma(X, X)$ with equality iff $X = J$.*

The proof of this lemma is very similar to the calculation done for the usual index lemma. (Roughly, choose a basis of normal Jacobi fields and write J, X in terms of it. Now compute the Riemann tensor term in the index form for X in terms of the derivatives using the Jacobi equation, and then use fundamental theorem of calculus, and so on.)
Now we can calculate and easily see that $\frac{d}{dt}\ln A(t,\theta) \leq \frac{d}{dt}\ln A_K$ and hence that $\frac{A}{A_K}$ is decreasing in t (at least when $t < t_{cut}$). Using the explicit expression for the Jacobi fields in normal coordinates, we see that the limit as $t \to 0$ is 1. Thus $A \leq A_K$. If we have equality for some $t = R$ and all θ, (in particular, we require R to be less than the injectivity radius at the point), then by monotonicity, $A = A_K$ for all $t \leq R$. Thus $J_i = X_i$. Now $\tilde{J}_i = sn_K(R)J_i(t)$ satisfy $\tilde{J}_i(0) = 0$ and $\tilde{J}'_i(0) = e_i$. Using geodesic normal coordinates, we see (as usual) that the metric is precisely that of the space-form and hence $B(p, R, g)$ is isometric to $B(p_K, R, g_K)$. ☐

Suppose $r < inj_p$. Now $V_r = \int_0^r a(t)dt$ where $a(t) = \int_{S^{n-1}} A(t,\theta)d\sigma$ and likewise $V_K = \int_0^r a_K(t)dt$. Now if we consider $f(r_1, r_2) = \frac{\int_{r_1}^{r_2} a(t)}{\int_{r_1}^{r_2} a_K(t)dt}$, then $f(r_1, r_2)$ is decreasing in r_1, r_2: Indeed, $\int_{r_1}^{r_2} a_K(t)dt \int_s^{r_2} a(t)dt \leq \int_{r_1}^{r_2} a(t)dt \int_s^{r_2} a_K(t)dt$ and hence $\partial_{r_1} f \leq 0$. Likewise for r_2.
If equality holds, then $V_r = V_K(r)$ for all $r \leq R$. Upon differentiation, we see that $A = A_K$ for all $r < R$. Thus using exponential coordinates, we see that if $r < inj_p$,

then $B(p,r)$ is isometric to $B_K(p_K,r)$ (note that the diameter is $\leq \frac{\pi}{\sqrt{K}}$ if $K > 0$ by Bonnet-Myer).
For the "global" statement, we need something about the cut locus. To be continued.........