1 Recap

1. Proved a "local" Bishop-Gromov.

2 Comparison geometry

2.1 Bishop-Gromov volume comparison and rigidity

Lemma 2.1. *If $Ricc \geq (n-1)Kg$, then for all $t < t_{cut}(\theta)$,*

$$\frac{d}{dt}\ln A(t,\theta) \leq \frac{d}{dt}\ln A_K(t,\theta) = (n-1)\frac{sn'_K}{sn_K}. \tag{1}$$

Thus $A(t,\theta) \leq A_K(t)$. Moreover, equality holds for some $t = R$ and all θ iff $B(p,R)$ is isometric to $B_K(R)$.

Using this lemma, we can try to prove Bishop-Gromov, except that we need to known something about the cut locus:

Theorem 1. *Let (M,g) be a connected complete Riemannian manifold.*

1. *The cut time from the unit tangent bundle (that is, the subset of TM consisting of unit vectors) to $(0,\infty]$ is continuous.*
2. *The cut locus of a point p is a closed subset measure zero. That is, away from a set of measure zero, the exponential map is a diffeomorphism (and the distance function is smooth).*

Proof. 1. Suppose $(p,v) \in UTM$ and $(p_i,v_i) \to (p,v)$. Let $b = \liminf c_i = t_{cut}(p_i,v_i)$ and $c = \limsup c_i$. We shall prove that $c \leq t_{cut}(p,v) \leq b$.
$c \leq t_{cut}(p,v)$: Suppose $c < \infty$. Then, upto a subsequence, $c_i \to c$ and γ_{v_i} is minimising on $[0,c_i]$. By continuity, $d_g(p,\exp_p(cv)) = \lim d_g(p_i,\exp_{p_i}(c_iv_i)) = \lim c_i = c$ and hence γ_v is minimising on $[0,c]$. Thus $t_{cut} \geq c$. If $c = \infty$, the same argument shows that γ_v is minimising on arbitrarily large intervals.
$t_{cut}(p,v) \leq b$: WLog, $b < \infty$. Again, passing to a subsequence, $c_i \to b$. So either $\gamma_v(c_i)$ is conjugate to p_i for infinitely many i or there is another geodesic σ_i for infinitely many indices. In the first, by continuity, $\gamma_v(b)$ is a critical point of the exponential map and hence $t_{cut} \leq b$. In the second case, near the limit, the exponential map is $1-1$ and hence since $v_i \to w$ (after passing to a subsequence), we have a distinct geodesic $\exp_p(tw)$ which means that $t_{cut} \leq b$.

2. There are two things to prove.
   (a) Closed: If $\exp_p((c_i = t_{cut}(p,v_i))v_i) \to w$ (for unit vectors v_i), then $d(p,\exp_p(c_iv_i)) = c_i \to c$ for some c up to a subsequence. Now up to a further subsequence, $v_i \to w$ and by continuity, $t_{cut}(p,v) = c$. Hence $w = \exp_p(cv)$.
   (b) Measure zero: Note that the cut locus of p is the image of the cut-locus C in the tangent space under $\exp_p$. If we prove that C has measure zero, we will be done. Now C is locally a graph over a part of a sphere of a continuous function t_{cut} and hence has measure zero.

□

Now $V_r = \int_0^r a(t)dt$ where $a(t) = \int_{S^{n-1}} A(t,\theta)d\sigma$ and likewise $V_K = \int_0^r a_K(t)dt$. Now if we consider $f(r_1, r_2) = \frac{\int_{r_1}^{r_2} a(t)}{\int_{r_1}^{r_2} a_K(t)dt}$, then $f(r_1, r_2)$ is decreasing in r_1, r_2: Indeed, $\int_{r_1}^{r_2} a_K(t)dt \int_s^{r_2} a(t)dt \leq \int_{r_1}^{r_2} a(t)dt \int_s^{r_2} a_K(t)dt$ and hence $\partial_{r_1} f \leq 0$. Likewise for r_2.
If equality holds, then $V_r = V_K(r)$ for all $r \leq R$. Upon differentiation, we see that $A = A_K$ for all $r < R$. Thus using exponential coordinates, we see that if $r < inj_p$, then $B(p, r)$ is isometric to $B_K(p_K, r)$ (note that the diameter is $\leq \frac{\pi}{\sqrt{K}}$ if $K > 0$ by Bonnet-Myer). For arbitrary r, using the Jacobi field expressions ($J_i = X_i$) we see that since they coincide, the sectional curvatures are the same. Thus if $K > 0$, and $R > dia(M)$, then by Killing-Hopf, M is a quotient of S^n and since the volumes are equal, M is S^n. □

2.2 Cheng's diameter rigidity theorem

Recall Cheng's theorem: Let (M, g) be a complete Riemannian manifold with $Ric \geq (n-1)Kg$ where $K > 0$. If $diam(M) = \frac{\pi}{\sqrt{K}}$, then M is isometric to a sphere.
By rescaling, assume that $K = 1$. Choose p, q in M (which we know is compact by Bonnet-Myers) such that $d(p, q) = \pi$. By the triangle inequality, $B(p, \frac{\pi}{2}) \cap B(q, \frac{\pi}{2}) = \phi$ and hence $Vol(M) \geq Vol(B(p, \frac{\pi}{2})) + Vol(B(q, \frac{\pi}{2}))$. By Bishop-Gromov, there is an $1 \geq \alpha = \frac{Vol(M)}{Vol(S^n_K)}$ such that $Vol(B(p, r)) \geq \alpha Vol(B_K(p_K, r))$. If we prove that $\alpha = 1$, then by the equality case for Bishop-Gromov, we will be done. Now we see using Bishop-Gromov to each of the balls that $Vol(B(p, \frac{\pi}{2})) \geq \frac{Vol(M)}{2}$ and $Vol(B(q, \frac{\pi}{2})) \geq \frac{Vol(M)}{2}$. Putting these together, we see that equality holds. Thus for all $r \in [\frac{\pi}{2}, \pi]$, $Vol(B(p, r)) = \alpha Vol(B_K(p_K, r))$ and likewise for q. Now if $r < \frac{\pi}{2}$, again, $Vol(M) \geq Vol(B(p, r)) + Vol(B(q, \pi - r)) \geq \alpha Vol(S^n) = Vol(M)$ and hence equality holds for all r. Taking $r \to 0$, we see that $\alpha = 1$. □

2.3 Synge's theorem

We recall that the second derivative of the length functional is $\int(\|V'\|^2 - Riem(V, \gamma', \gamma', V))$. If V is somehow chosen to be parallel, and if the sectional curvature is > 0, then the second derivative is strictly negative and hence our geodesic is not a minimiser. Somehow we want to leverage this observation. The first step is the observation that in every non-trivial free homotopy class there is a minimising closed geodesic. If we can use these two observations together, maybe we can conclude simple-connectedness somehow. To delve into more detail, how may one construct a variation field V such that say $\Gamma(s, t) = \exp_{\gamma(t)}(sV(t))$ is a variation by closed loops? One way is to try paralell transporting some vector $V(0)$ from the starting point of the geodesic loop under consideration. Unfortunately, $V(1)$ need not be equal to $V(0)$. So we need a parallel normal (why normal?) field such that $V(1) = V(0)$. Note that since parallel transport preserves inner products, the parallel transport map $P : T_pM \to T_pM$ has determinant ± 1, and every eigenvalue is ± 1. Thus, some parity (odd/even) will determine whether 1 occurs as an eigenvalue (other than for the trivial eigenspace spanned by $\gamma'(0)$) or not. Therefore, we expect some orientability to play a role. That is the content of Synge's theorem:

Theorem 2. *Let (M, g) be a compact Riemannian manifold with positive curvature. Then*

1. *if M is even dimensional and orientable, then M is simply connected.*
2. *if M is odd dimensional, it is orientable.*