1 Recap

1. Bishop-Gromov.
2. Cheng diameter rigidity.
3. Motivated the statement of Synge and reduced it to properties of the determinant of the parallel transport map, and to a study of orientability.

2 Comparison geometry

2.1 Synge's theorem

Theorem 1. *Let (M, g) be a compact Riemannian manifold with positive curvature. Then*

1. *if M is even dimensional and orientable, then M is simply connected.*
2. *if M is odd dimensional, it is orientable.*

Before we prove this theorem, we shall develop some theory about orientation (it is useful regardless of this particular theorem).
Firstly, given a manifold M, we can come up with a double-cover (not necessarily connected) $\tilde{M}$ called the orientation double cover of M: Consider the set $\tilde{M} = \cup_{p \in M}(p, o_p)$ where o_p is the two-element set of orientation classes on T_pM (recall that an orientation on a vector space is simply a choice of ordered basis and the equivalence relation is the determinant of the change of basis is 1). There is an obvious projection map $\pi : \tilde{M} \to M$. Using coordinate charts on M, we can make $\tilde{M}$ into a manifold, with π being a smooth covering map (how?). Also, $\tilde{M}$ is orientable (why?)

Lemma 2.1. *M is orientable iff $\tilde{M}$ is disconnected. Moreover, as a cover in this case, it is diffeomorphic to $M \times Z_2$.*

Proof.

1. M is orientable: Choose an orientation O. Consider the section $s : M \to \tilde{M}$ given by $s(p) = O(p)$. The image of s is connected. The image of $\tilde{s} = -O(p)$ is also connected. These two images do not intersect. Hence $\tilde{M}$ is disconnected. Moreover, $M \times \{-1, 1\} \to \tilde{M}$ given by $f(p, x) = (p, xO(p))$ is a diffeomorphism.
2. $\tilde{M}$ is disconnected: Let C be the connected component of (p, o). We claim that
   (a) $\pi(C) = M$: Suppose $q \notin \pi(C)$. Then consider a path $\gamma(t)$ joining p and q. It lifts upstairs starting at (p, o) to (q, o_q). A contradiction.
   (b) C does not have $(p, -o)$: Indeed, if it did, then consider a path $\gamma(t)$ between (p, o) and $(p, -o)$. Then we claim that for any point $(q, o_q) \in C$, it can be connected to $(q, -o_q)$ and hence $\tilde{M}$ is actually connected. Take a path $s(t)$ downstairs connecting q and p. Now first follow $s(t)$ and then $\pi(\gamma(t))$. This lifts upstairs uniquely to a path connecting (q, o_q) to $(p, -o)$. Now come back to q downstairs via the reverse of $s(t)$. The lift upstairs cannot end at (q, o_q) by uniqueness of the lift. Hence it ends at $(q, -o_q)$.

□

As a corollary, we see that (M, g) is non-orientable iff for every point $p \in M$, there exists a loop from p such that $\det(P) = -1$ (how?).
Now we can prove Synge's theorem:

1. If M is orientable, then $\det(P) = 1$ for a minimising geodesic loop in a non-trivial free homotopy class (if $\pi_1(M) \neq \{1\}$). Since $P(\gamma'(0)) = \gamma'(0)$, if M is even-dimensional, the orthogonal complement is odd-dimensional and since $\det(P) = 1$, 1 occurs as an eigenvalue. We are done by the above argument.

2. If M is odd-dimensional and non-orientable, note that since it has a double cover, $\pi_1(M) \neq 0$. Moreover, there exists a loop in a non-trivial homotopy class such that $\det(P) = -1$.

As a corollary, if M is compact, even-dimensional, non-orientable, and has positive curvature, then $\pi_1(M) = \mathbb{Z}_2$ (why?). Thus $\mathbb{RP}^2 \times \mathbb{RP}^2$ cannot admit a metric of positive sectional curvature. However, it is not yet known whether $S^2 \times S^2$ can or not (the famous Hopf conjecture).

2.2 Preissmann's theorem

We have mostly dealt with positive curvature. Preissmann's theorem deals with negative sectional curvature. As in the case of Synge, we want to know something about the fundamental group. Any subgroup H of π_1 corresponds to a group of deck transformations of the universal cover. (Also, every deck transformation is an isometry of the pullback metric of the universal cover. Why?) Now since we know closed minimising geodesics exist in every free homotopy class (and this fact helps with the fundamental group as in Synge), it is natural to ask which geodesics are left invariant by a deck transformation $\psi \in H$ (if there are several for each element, maybe H is not too large?). Also, we know that if $\tilde{\gamma}$ is a geodesic which is a reparametrisation of γ, then these two parametrisations are related by an affine map. By the isometry property, $\tilde{\gamma}(t) = \gamma(t+a)$ (or a backward parametrisation). We can hope that for every element of H, there is a geodesic γ and $\psi(\gamma(t)) = \gamma(t+a)$ for a unique a. Then we can associate a real number a to every element of H. Hopefully, this means that H is a cyclic group. This flimsy motivation allows us to conjecture and prove the following result of Preissmann.

Theorem 2. *If (M, g) is a compact (connected as always) Riemannian manifold with $\sec < 0$, then every nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

The "motivation" above motivates the following definition: Let (M, g) be a complete manifold and $\phi : M \to M$ be an isometry. A geodesic γ is said to be an axis for ϕ if $\phi(\gamma(t)) = \gamma(t+a)\ \forall\ t$. An isometry is said to be axial/translation along γ if it does not have fixed points and if it has an axis γ. (As an example, every translation of Euclidean space has an axis.)