1 Recap

1. Orientation double cover
2. Proof of Synge
3. Statement of Preissmann and definition of axis.

2 Comparison geometry

2.1 Preissmann's theorem

Theorem 1. *If (M, g) is a compact (connected as always) Riemannian manifold with $sec < 0$, then every nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

Proof. Let H be an Abelian subgroup. Let $\alpha \in H$. There is a deck transformation f_α corresponding to α defined as follows: Take any loop γ in M based at $p = \pi(\tilde{p})$ that is in the based-homotopy class of α. Let $\tilde{\gamma}$ be the unique lift of γ. The endpoint of $\tilde{\gamma}$ is independent of γ! (This is a known fact.) Define $f_\alpha(\tilde{p})$ as this endpoint. It is also well-known that this f_α is a deck transformation (and $f_{\beta\alpha} = f_\beta \circ f_\alpha$). Any deck transformation is an isometry. Non-trivial deck transformations do not have fixed points (why?).
Now if $\alpha \neq 0 \in H$, let γ be any closed geodesic in the free homotopy class determined by α and $\tilde{\gamma}$ be a lift. The first observation is that:

Lemma 2.1. *f_α is a translation along $\tilde{\gamma}$.*

Proof. We can easily show that $f_\alpha(\tilde{\gamma}(t))$ (which is a geodesic) has the same first order initial data as the geodesic $\tilde{\gamma}(t+l)$ (where l is the length of γ) and hence we are done. □

Since $f_{\alpha\beta} = f_{\beta\alpha}$, $f_\beta(\tilde{\gamma}(t)) = f_\alpha(f_\beta(\tilde{\gamma}))$. Now

Lemma 2.2. *Suppose (M, g) has negative sec, then any translation $f : \tilde{M} \to \tilde{M}$ fixes only one geodesic.*

Proof. Suppose there are two geodesics σ_1, σ_2 such that f fixes both. Firstly, they do not intersect. Indeed, if they do, then since f has no fixed point, there are at least two points in the intersection. But the exponential map is a diffeomorphism and hence this is a contradiction.
Now choose $q_i \in \sigma_i$ and let σ_3 be the minimal geodesic joining them. Consider the geodesic quadrilateral $q_1, q_2, f(q_2), f(q_1)$. Since f is an isometry, the interior angles at $q_i, f(q_i)$ add up to π. But splitting is quadrilateral into two triangles and using the "cosine rule inequality", we see that the angles could not have added up to 2π. (How exactly is this a contradiction? Note that using the triangle inequality on the sphere, $\langle ABD \leq \langle ABC + \langle CBD$) □

As a corollary, if one has negative sectional curvature, there is a unique closed geodesic in every free homotopy class.
Thus $\tilde{\gamma}$ is the unique geodesic fixed by all elements of H. Now since it is a unit-speed

geodesic, either $f_\beta(\tilde{\gamma}(t)) = \tilde{\gamma}(t + a = t_\beta)$ or $\tilde{\gamma}(-t + a)$. The latter cannot occur because $\tilde{\gamma}(a/2)$ will be a fixed point.
Now we can define $\phi : H \to \mathbb{R}$ as $\phi(\alpha) = t_\alpha$. Here are a few properties of ϕ:

1. ϕ is a group homomorphism: $\phi(\alpha\beta) = t_{\alpha\beta}$. Now $f_{\alpha\beta}(\tilde{\gamma}(t)) = f_\alpha\tilde{\gamma}(t + t_\beta) = \tilde{\gamma}(t + t_\beta + t_\alpha)$. Thus $t_{\alpha\beta} = t_\alpha + t_\beta$.

2. ϕ is injective: Indeed, if $\phi(\alpha) = \phi(\beta)$, then $f_\beta(\tilde{\gamma}(t)) = f_\alpha(\tilde{\gamma}(t))$ for all t. This means that $f_{\beta\alpha^{-1}}(\tilde{\gamma}(0)) = \tilde{\gamma}(0)$ and hence $f_\beta = f_\alpha$.

3. The image of ϕ is NOT dense: Choose a evenly covered geodesically normal ball of radius r around $p = \pi(\tilde{\gamma}(0))$. For each $\beta \neq e$, $f_\beta(\tilde{p}_0) \notin U_0$ where U_0 is one containing $\tilde{p}_0$ (because in a normal ball, all geodesics from the centre are radial and hence do not return to the centre). Thus $|t_\beta| \geq inj(M)$.

As a consequence, H is an additive subgroup of $\mathbb{R}$ that is not dense. Hence it is infinitely cyclic. (Indeed, let b be the smallest positive element of H. Then there is no element between b and $2b$.) □