1 Recap

1. Course logistics and motivation.

2. Curvature of plane curves.

3. Hopf's rotation theorem.

2 Further motivation through surfaces in space

Before defining Riemannian manifolds, curvature, etc, we shall present the case of surfaces quickly (without going into details) for motivation.

A smooth surface without boundary $S \subset \mathbb{R}^3$ is a subset with the induced topology such that near every point p, there is a coordinate parametrisation, i.e., a smooth map from an open set $U \subset \mathbb{R}^2$ $\vec{r}(u,v) : U \to S \subset \mathbb{R}^3$ that is $1-1$, $\vec{r}(U)$ is a relatively open neighbourhood of p, and $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}$ are linearly independent. An example of a surface is the sphere $x^2+y^2+z^2=1$ (using the implicit function theorem).

Any two such parametrisations are reparametrisations of each other using the inverse function theorem. Moreover, the inverse $\vec{r}^{-1}$ can be extended as a smooth map in a neighbourhood of p in $\mathbb{R}^3$. If $\gamma(t)$ is a path on the surface whose image is in $\vec{r}(U)$, then it corresponds to a path $u(t), v(t)$ and hence by the chain rule, $\gamma'(t)$ is a linear combination of $\vec{r}_u, \vec{r}_v$. We define the tangent space T_pS at $p \in S$ to be the subspace of $\mathbb{R}^3$ spanned by $\vec{r}_u, \vec{r}_v$ at p. A normal N is $\vec{r}_u \times \vec{r}_v \neq 0$.

Assume that S is connected. How can one define the distance between two points on a surface? The simplest way is to take the infimum of lengths of continuous piecewise C^1 paths lying on S connecting p and q. (Is this infimum always a minimum?) The infinitesimal properties of this distance is captured by inner product on the tangent space. Choosing a local basis as $\vec{r}_u, \vec{r}_v$, we see that the matrix is given by $G = \begin{bmatrix} r_u^2 & r_u.r_v \\ r_u.r_v & r_v^2 \end{bmatrix}$. This matrix is sometimes called the first fundamental form of the surface (and will later be called the induced Riemannian metric in local coordinates). Note that if we change the coordinates by a reparametrisation, G does change but only using the derivative matrix of the reparametrisation (by a similarity transformation). For a sphere, if we choose θ, ϕ, then G is diagonal with entries $1, \sin^2(\theta)$.

If we could find a to-scale map of a part of the earth, then in particular, using the coordinates provided by the "map", the matrix G is identity. That is, can we always find a parametrisation such that G is identity? Gauss identified a quantity (now called the Gaussian curvature) that is invariant under reparametrisation and turns out to be non-zero for the earth but zero for a plane (piece of paper) and hence this is not possible. This quantity captures the *intrinsic* curvature of the earth (as opposed to folding a piece of paper and making it look curved).

There are two ways to approach this intrinsic curvature business:

1. Through curves and their curvature: Consider a unit-speed curve γ on S through p. The naive definition of curvature would be $\|a(s)\|$ but the problem is that the acceleration need not point along the surface and that part is not relevant whilst considering the curvature of the surface itself because the tangential part measures

how much the curve curves within the surface. Now define $\kappa_\gamma = \langle a, N\rangle$. This quantity obvious varies with the curve. Define $K = \max \kappa_\gamma \min \kappa_\gamma$. This seems like the determinant of a certain matrix. Indeed, $\kappa_\gamma = -\langle v, \frac{dN}{ds}\rangle = -\langle v, DNv\rangle$. The determinant of this bilinear form (DN is called the shape operator or the Weingarten map) is K. Note that the direction of the normal makes no difference to K (and perhaps is a clue to why K is defined the way it is). In terms of the first fundamental form, we can write a formula for K using only G and its derivatives (two suffice). This will be a HW problem. One can see how this formula changes under reparametrisation (it doesn't) - HW and prove the Theorem Egregium. (Does the trace of the shape operator have a meaning? It is called the mean curvature and is related to soap bubbles!)

2. Through parallel transport: Is the sum of angles of a "triangle" (a triangle is one whose sides are "lines", that is, distance-minimising curves) 180 degrees on the earth? Take one with a vertex at the north pole and two on the equator. The sum is obviously larger! (note that if we consider smaller and smaller triangles, it seems that they get closer and closer to Euclidean ones and hence the sum of angles gets close to 180 degrees. In fact, this fact is intimately related to K!) So what goes wrong in the "usual" proof? The usual proof moves one side parallel to itself to a vertex. So can we at least "parallel transport" tangent vectors (forget the entire side itself!) along a curve? That is, what does it mean for a vector field Y to be parallel along a curve γ? Naively, $\frac{dY}{dt} = DYv = 0$ but that is problematic because the only relevant part is the tangential component. So we define the "covariant derivative" $\frac{DY}{dt} = \frac{dY}{dt} - \langle \frac{dY}{dt}, N\rangle N = \frac{dY}{dt} + \langle \frac{dN}{dt}, Y\rangle N$. Parallel transport would mean solving $\frac{DY}{dt} = 0$ with $Y(0) = Y_0$. (Later we shall see that by ODE theory, we can indeed solve this equation.) We can hope that a length minimising curve "follows its nose", that is, $\frac{Dv}{dt} = 0$ and indeed we shall see later that this is true. We can now see what happens when we take any infinitesimal parallelogram $\vec{r}(u, v), \vec{r}(u + du, v), \ldots$ and parallel transport a tangent vector around it. A calculation shows that the change (of angle, because the length remains the same) is actually K times the area of this parallelogram. This is a case of the local Gauss-Bonnet theorem.