1 Recap

1. Definition of surfaces.

2. First fundamental form.

3. Curvature using the shape operator and using parallel transport.

2 Review of Manifolds, Tangent bundle, Vector fields, Lie bracket, etc

2.1 Manifolds

We want to study the curvature of a generalisation of surfaces. The correct generalisation is a manifold (with or without boundary) equipped with a Riemannian metric. Recall the two (eventually equivalent) definitions of a smooth n-dimensional manifold-without-boundary M:

1. Extrinsic (as an embedded submanifold of $\mathbb{R}^N$): $M \subset \mathbb{R}^N$ with the induced topology such that every point has a neighbourhood V that can be parametrised using an open subset $U \subset \mathbb{R}^n$ as $\phi : U \to V = \phi(U)$ where ϕ is smooth into $\mathbb{R}^n$, $1-1$, and $D\phi$ is $1-1$. (As a consequence of the constant rank theorem, ϕ^{-1} can be extended to a smooth map from a neighbourhood of U.)

2. Intrinsic: M is a Hausdorff second countable space equipped with a maximal smooth atlas. A smooth atlas $\mathcal{A}$ is an open cover U_α of M by sets that are homeomorphic to open subsets $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ such that $\phi_\alpha \cap \phi_\beta^{-1}$ are diffeomorphisms. Two atlases are compatible if their union is a smooth atlas. A maximal smooth atlas is a smooth atlas such that any smooth atlas compatible with it is already contained in it. Every smooth atlas is contained in a unique maximal one (and hence it is enough to specify a smooth atlas without worrying about the adjective maximal).

A sphere is a manifold (either using stereographic projections or the implicit function theorem). A smooth map $f : M \to N$ is a continuous map such that in coordinates it is smooth. f is said to be an immersion if Df in coordinates is $1-1$ (if it is so in one pair of coordinate charts, it is so in any other pair). A $1-1$ immersion that is a homeomorphism to its image is called an embedding. An embedded submanifold (or simply, a submanifold) is a subset $S \subset M$ which is a smooth manifold in its own right such that the inclusion map is an embedding. A diffeomorphism is a smooth homeomorphism whose inverse is also smooth. The aim of differential topology is to classify manifolds upto diffeomorphism. This aim has been realised for 1 and 2-dimensional manifolds. It cannot be realised for dimensions ≥ 4 in a sense (because of the existence of undecidable problems).
The Whitney embedding theorem states that every smooth n-dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$ (and thus the intrinsic and extrinsic definitions are equivalent).

A regular value of $f : M \to N$ is a point $c \in N$ such that for every point $p \in f^{-1}(c)$, Df is surjective (in any pair of charts). $f^{-1}(c)$ is then a submanifold of dimension $dim(M) - dim(N)$ (it could be empty too). A critical value is a point that is not a regular value. The set of critical values in N has measure zero in N (which means it can be covered with countably many charts such that in each chart, the set has measure zero).
A smooth manifold-with-boundary is a Hausdorff second countable space such that every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ with the transition maps being smooth. The boundary ∂M is the collection of points that do not have any neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. It turns out to be a submanifold of dimension $n - 1$. Moreover, its points are inverse images of ∂H^n. If c is a regular value of $f : M \to \mathbb{R}$ and $f > c$ and $f = c$ are non-empty, then $f \geq c$ is a smooth manifold-with-boundary such that its boundary is $f^{-1}(c)$.

Every smooth manifold has a compact exhaustion. It is also paracompact. Moreover, it has a partition-of-unity subordinate to a given open cover. If we allow ourselves a locally finite refinement, then the partition-of-unity can be chosen to have compact support.

A manifold (with or without boundary) is said to be orientable if it has an oriented atlas, i.e., an atlas such that $\det(D(\phi_\alpha \cap \phi_\beta^{-1})) > 0$ for all $\alpha \neq \beta$. It is said to be oriented if it is provided with an oriented atlas. Two atlases are said to be orientation compatible if their union is oriented. This defines an equivalence relation between atlases. The equivalence classes are called orientations and there are exactly two of them if the manifold is connected (which we will assume wlog from now onwards).

A covering space Y (in the sense of topology) of a manifold X as a natural smooth structure. Indeed, if U_α is a coordinate chart contained in an evenly covered neighbourhood, then $\pi^{-1}(U_\alpha) = \cup_i V_{\alpha,i}$ with π being a homeomorphism from $V_{\alpha,i}$ to U_α for each i. Now we get charts on the cover such that π is a smooth immersion.

2.2 Tangent bundle and vector bundles

Note that it is hard to produce examples of diffeomorphisms. One way to try to produce them is to imagine a fluid is flowing along the manifold and follow it for some time. To this end, we need to define a "smoothly varying collection of tangent vectors". Firstly, one needs to make sense of a tangent vector. The tangent space T_pM is defined in several (equivalent ways):

1. Derivations on the ring of smooth functions: A derivation $D : \mathcal{C}^\infty(M) \to \mathbb{R}$ is a linear map such that $D(fg) = f(p)Dg + g(p)Df$. The collection of derivations forms a vector space T_pM. It turns out that this vector space is n-dimensional and $T_pM = T_pU$ for any coordinate chart U canonically, and $\frac{\partial}{\partial x^i}|_p$ span this space.

2. Equivalence classes of curves: Let $\gamma(t) : (-\epsilon, \epsilon) \to M$ be a smooth map such that $\gamma(0) = p$. Define an equivalence relation between two such maps $\gamma_1 \sim \gamma_2$ if $\gamma_1'(0) = \gamma_2'(0)$ in one (and hence any) coordinate chart. The set of equivalence classes is T_pM with addition defined as coordinate-wise addition (in any chart).

3. Physicist's definition: $T_pM = \cup_\alpha \mathbb{R}^n_{\alpha\ p\in U_\alpha} / (v_\alpha \sim v_\beta\ if\ \frac{\partial x^i_\alpha}{\partial x^j_\beta}(p) v^j_\beta = v^i_\alpha)$.

Given a smooth map $f : M \to N$, the derivative/pushforward at p is $f_* : T_pM \to T_pN$ defined as $(f_*D)(g) = D(g \circ f)$ using derivations or $f_*v = Dfv$ using the physicist's definition/local coordinates.
Next, a smoothly varying vector field (in the physicist definition) is simply a collection of smooth functions $X_\alpha : U_\alpha \to \mathbb{R}^n$ that transform correctly, i.e., $X^i_\alpha = \frac{\partial x^i_\alpha}{\partial x^j_\beta} X^j_\beta$. Ideally we would like to say that $TM = \cup_p T_pM$ is a manifold such that a smoothly varying vector field is a smooth map from M to TM commuting with the obvious projection. To this end, we need to make TM into a manifold. Cover M with a countable atlas. Now consider the bijection $T_\alpha : U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha) \subset TM$ given by $T_\alpha(p, \vec{v}) = v^i \frac{\partial}{\partial x^i_\alpha}(p) \in T_pM$. Consider a countable basis of U_α and define a topology by using a countable basis of TM using T_α. This topology is Hausdorff and (obviously) second countable. It is locally Euclidean too (using T_α^{-1}) and the transition maps are diffeomorphisms of a very particular type: $(p, \vec{v}) \to (p, w^i = \frac{\partial x^i}{\partial y^j} v^j)$ which preserve the vector space structure of each T_pM.

We can generalise this construction to get a smoothly varying family of vector spaces. A smooth real vector bundle V of rank r over M is an $n + r$-dimensional manifold equipped with a smooth surjective projection map $\pi : V \to M$ such that $\pi^{-1}(p) = V_p$ (the "fibre") is a real r-dimensional vector space for every p, and V is locally trivial, i.e., there exists a diffeomorphism $\pi^{-1}(U_p) \to U_p \times \mathbb{R}^r$ that commutes with projections and is a linear isomorphism on each fibre.
A smooth section $s : M \to V$ of a vector bundle is a smooth function such that $\pi(s(p)) = p$. Having a smooth local collection of r sections s_i such that $s_i(p)$ forms a basis for V_p is equivalent to being locally trivial. Such sections are said to form a local frame/trivialisation.
A vector bundle morphism $T : V \to W$ is a smooth map commuting with projections that is linear on each fibre. Other than TM, there are several natural vector bundles associated to M. The cotangent bundle $T^*M = \cup_p T^*_pM$ equipped with $T^*_\alpha(p, \vec{\omega}) = \omega_i \left(\frac{\partial}{\partial x^i}\right)^*$ give T^*M a smooth vector bundle structure. If V, W are vector bundles, then $V \oplus W$ and $V \otimes W$ are defined using local trivialising sections as $s_i \oplus t_j$ and $s_i \otimes t_j$ being the local trivialising sections. In this way, we can define tensor bundles. If $f : M \to \mathbb{R}$ is a smooth function, then df is a smooth 1-form defined as $df_p(X_p) = X_p(f)$. In local coordinates, $df = \frac{\partial f}{\partial x^i} \left(\frac{\partial}{\partial x^i}\right)^*$.