1 Recap

1. Review of manifolds (with and without boundary), submanifolds, Whitney embedding, and regular values.

2. Tangent spaces and the tangent bundle (various definitions), vector bundles, and the cotanget bundle.

2 Review of Manifolds, Tangent bundle, Vector fields, Lie bracket, etc

2.1 Forms, wedge product, and the exterior derivative

Multilinear maps are called tensors and multilinear maps factor as linear maps from the tensor product. Multilinear maps have bases $\epsilon_{i^1} \otimes \ldots$. Among tensors we have symmetric and skew-symmetric/alternating ones defined in the usual manner. The alternating ones are also called forms. They have a basis $\epsilon^I(v_1, \ldots, v_k) = \det(\epsilon^i(v_j))$. Given a $(0,k)$-tensor T, we can produce a k-form using the Alt construction as follows: $Alt(T)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma(1)}, \ldots)$. Using this construction we can generalise the cross product to the wedge product of forms as $\omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta)$. Now $\epsilon^{IJ} = \epsilon^I \wedge \epsilon^J$ and this product is associative and bilinear. Also, $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. If ω is a 0-form (a number), then $\omega \wedge \eta = \omega\eta$.
We can construct the vector bundle of k-forms, the sections of which are called k-form fields, or differential k-forms, or abusing terminology, simply k-forms again. The wedge product extends to form fields. We can define the pullback of form fields using a smooth map $f : M \to N$ as $(f^*\omega)_p(X_p) = \omega_{f(p)}(f_* X_p)$.
Now we define the exterior derivative of a form field in $\mathbb{R}^n$: If $\omega = \omega_i \epsilon^I$, then $d\omega = d\omega_i \wedge \epsilon^I = \frac{\partial \omega_i}{\partial x^j} \epsilon^{jI}$. In particular, $df = \frac{\partial f}{\partial x^i} dx^i$ and $\epsilon^I = dx^{i_1} \wedge dx^{i_2} \ldots$. The pullback commutes with d and $d^2 = 0$. Now $f^*\omega = \omega_I \circ f df^{i_1} \wedge \ldots$. This exterior derivative extends to manifolds too.
Forms ω such that $d\omega = 0$ are closed and if $\omega = d\eta$, they are called exact. Of course exact forms are closed but the converse is false. It is true for star-shaped domains in $\mathbb{R}^n$ (Poincaré lemma). The failure is measured by the de Rham cohomology vector spaces: $H^k(M)$ is closed k-forms that are quotiented out by exact ones. $H^0(M)$ is the number of connected components and $H^k(M) = 0$ if $k > n$. It turns out that if a manifold is not compact or not orientable, then $H^n(M) = 0$. If M is compact and orientable, then $H^n(M) = \mathbb{R}$.

2.2 Vector fields, flows, and the Lie bracket

A smooth vector field is a section of the tangent bundle TM, i.e., $X :\to TM$ is smooth and commutes with the projection. Locally, it is a local collection of smooth functions associated to every chart X^i (where $X = X^i \frac{\partial}{\partial x^i}$ in Einstein summation - the repeated indices are summed over) such that when we change coordinates, $X^\alpha = \frac{\partial y^\alpha}{\partial x^i} X^i$. Alternatively, it is an $\mathbb{R}$-linear map $C^\infty(M) \to C^\infty(M)$ such that $X(fg) = fX(g) + gX(f)$.

Indeed, it is easy to see that a vector field produces such a map. Given such a map, if x^i are coordinates, and ρ a bump function that is identically 1 in a neighbourhood of p, then $X(p) = X(x^i\rho)\frac{\partial}{\partial x^i}$ (why is this valid?). Given a smooth vector field X on a compact manifold (without boundary), and a point $p \in M$, there exists a unique integral curve $\gamma_p : \mathbb{R} \to M$ through p, i.e., $\gamma(0) = p, X(\gamma_p(t)) = (\gamma_p)_*\partial_t$ (written as $X(\gamma_p(t)) = d\gamma_p/dt$. Moreover, γ depends smoothly on p, t (taken jointly) and if $F(p,t) = \gamma_p(t)$, then for every t, F is a diffeomorphism and it satisfies $F(p, t+s) = F(F(p,t),s) = F(F(p,s),t)$ and $F(F(p,t),-t) = p$. It is a one-parameter group of diffeomorphisms. These facts can be proven using the theory of ODE.

Using bump functions, one can construct several non-trivial examples of vector fields. In fact, if $dim(M) \geq 2$, then any two points can be exchanged by means of diffeomorphisms.

Locally, $\frac{\partial}{\partial x^1}$ is a vector field. If $X(p) \neq 0$, the converse is also true, that is, $X = \frac{\partial}{\partial x^1}$ locally for an appropriate choice of coordinate x^1 (again by flowing using ODE theory). If we have two vector fields X, Y that are linearly independent at p (and hence nearby), it is *not* true that $X = \partial_1$ and $Y = \partial_2$. The reason is that $[\partial_1, \partial_2]f = 0$ whereas $X(Y(f)) - Y(X(f)) =: [X,Y](f)$ is not necessarily zero. The vector field $[X,Y] = XY - YX$ is called the commutator or the Lie bracket of X and Y. It obeys skew-symmetry, bilinearity, and the Jacobi identity. This is true for any number of vector fields. (A special case of the Frobenius theorem.)

2.3 Integration and Stokes' theorem

A connected manifold-with-boundary M is said to be orientable if there exists a nowhere vanishing top form ω. Two such top forms are said to define the same orientation if they are proportional by a positive function. (Otherwise, they are of opposite orientation.) An atlas is said to be orientation compatible with ω if $\omega(\partial_1, \partial_2, \ldots, \partial_n) > 0$ for all charts. If $dim(M) \geq 2$, then an atlas is orientation compatible with some form if $\det(\frac{\partial y^i}{\partial x^j}) > 0$ for all pairs. Two atlases are said to define the same orientation if their union is an oriented atlas.

If $f : \mathbb{R}^n \to \mathbb{R}$ has c as a regular value (with $f^{-1}(c) \neq \phi$), then $f^{-1}(c)$ is an orientable (sub)manifold (without boundary). Indeed, take $N = \nabla f$ and consider the top form $\omega(X_1, \ldots, X_{n-1}) = dx^1 \wedge dx^2 \ldots dx^n(N, X_1, \ldots) = \det(NX_1 \ldots)$. If M has a boundary ∂M, then the boundary inherits an orientation from M by choosing an "outward pointing vector N". Alternatively, one can simply choose boundary charts and restrict them to ∂M. These orientations are not the same! They agree only with the dimension of M is even! (We choose the former to be the "correct" induced orientation.) There are non-orientable manifolds too! (like $\mathbb{RP}^2$).

Given an oriented manifold M (with or without boundary) and a smooth top form ω, $\int_M \omega := \sum_i \int_M \rho_i\omega$ (for any partition of unity ρ_i) where $\int_M \omega$ is defined when ω is compactly supported in an oriented chart as $\int_{\mathbb{R}^n} (\phi^{-1})^*(\omega) = \int_{\mathbb{R}^n} f \circ \phi$ in the sense of Lebesgue (where $\omega = f dx^1 \wedge dx^2 \ldots$). The change of variables formula shows that this definition is well-defined. It turns out that to calculate using this definition, it is enough to use one chart that covers the manifold-upto-measure zero (measure zero on a manifold simply means that around every point there is a chart in which the set has measure zero). For instance, one can calculate an integral over a sphere using stereographic coordinates or latitude and longitude.

Stokes theorem: If M is oriented and compact, and ∂M has the induced orientation if $dim(M)$ is even and the opposite one if $dim(M)$ is odd, then $\int_M d\omega = \int_{\partial M} \omega$ if ω is an $n-1$-form. In particular, if there is no boundary, then $\int_M d\omega = 0$.