1 Recap

1. Review of forms, exterior derivative, etc.
2. Review of vector fields and flows.
3. Review of integration and Stokes' theorem.

2 Riemannian metrics

Let M be a smooth manifold (with or without boundary). A Riemannian metric g is a smooth section of $T^*M \otimes T^*M$ that is symmetric, i.e., $g(X,Y) = g(Y,X)$ and is positive-definite, i.e., $g(X,X) > 0$ if $X \neq 0$. It is basically a smoothly varying family of inner products. Locally, it is $g = g_{ij}dx^i \otimes dx^j$ where the smooth local matrix-valued function g_{ij} is a symmetric positive-definite matrix. $g(X,Y) = X^T gY$ locally.
Proposition: Riemannian metrics exist on any manifold (with or without boundary).
Proof: Let ρ_i be a partition-of-unity subordinate to a collection of charts. Define $g = \sum_i \rho_i \sum_j dx_i^j \otimes dx_i^j$. This defines a Riemannian metric essentially because the sum of positive-definite matrices is positive-definite. □

A Lorentzian metric is a smooth symmetric section with signature $(n-1,1)$. Not every manifold admits a Lorentzian metric! (Indeed, the sum of such matrices is not of the same type in general) It turns out that either the manifold has to be noncompact or must satisfy $0 = \sum_i (-1)^i dim(H^i(M))$. We will not deal with Lorentzian geometry in this course. See O'Neill's book if interested. Some theorems in Lorentzian geometry (intimately connected to General Relativity) like the Hawking-Penrose singularity theorems have analogues that are very different in flavour in Riemannian geometry (like the Bonnet-Myers theorem).

An isometry $\phi : (M,g) \to (N,h)$ is a diffeomorphism such that $\phi^*h = g$, i.e., $h_{\phi(p)}(\phi_* X, \phi_* Y) = g_p(X,Y)$. The simplest example of a Riemannian manifold is of course the Euclidean metric $g = \sum_i dx^i \otimes dx^i$. Clearly translations and orthogonal transformations are isometries. (It turns out that upto composition, they are the only ones.) It turns out (a difficult theorem of Myers-Steenrod) that the isometry group of a connected manifold is a finite-dimensional Lie group acting smoothly on M. Recall that a Lie group G is a group that is also a smooth manifold such that the group operations are smooth. For instance, $GL(n)$ is a Lie group. So is $SL(n)$ (why?) and so on. A Lie group G is said to act smoothly on M if the action $G \times M \to M$ is smooth. For instance, $GL(n)$ acts smoothly on $\mathbb{R}^n$ in the usual way. More non-trivially, $(\mathbb{R},+)$ acts smoothly on a compact manifold equipped with a vector field as $(t,p) \to F(t,p)$ where F is the flow.
Here are a few ways of constructing new metrics from old ones:

1. Conformal change: Let g be a Riemannian metric on M and let $f : M \to \mathbb{R}_+$ be a smooth positive function. Then fg is another Riemannian metric on M. Infinitesimally, it preserves "angles" but not lengths. It is an example of a conformal change. (Proceeding further, a conformal diffeomorphism $\phi : (M,g) \to (N,h)$ is one such that $\phi^*h = fg$ for some positive function f. It turns out (shockingly

enough) that *locally*, any metric on a surface is conformal to a Euclidean metric. (Such coordinates are called isothermal coordinates because the coordinate functions are harmonic, i.e., steady states of the heat equation.) Moreover, here is an important example: Consider $\mathbb{H}^n$. It has a metric (called the hyperbolic metric): $g = \frac{g_{Euc}}{(x^n)^2}$.

2. Riemannian submanifolds: Let (N, h) be a Riemannian manifold with or without boundary. Let M be a smooth manifold with or without boundary and $F : M \to N$ be a smooth map. Then F^*h is a Riemannian metric on M iff F is an immersion. Indeed, if F is an immersion, it is easy to see this fact. If $g = F^*h$ is a Riemannian metric and if $F_*X = 0$, then $g(X, X) = h(F_*X, F_*X) = 0$ and hence $X = 0$. Thus F is an immersion. $\square$
   In particular, if M is an embedded submanifold (with or without boundary), then we have the *induced* Riemannian metric on M. Thus the n-sphere S^n has an induced Riemannian metric. If we consider a local part of it given as a graph $x^n = \sqrt{1-(x^1)^2\ldots}$, then

$$g = \sum_{i=1}^{n-1} dx^i \otimes dx^i + d\sqrt{1-(x^1)^2\ldots} \otimes d\sqrt{1-(x^1)^2\ldots} = \sum_{i=1}^{n-1} dx^i \otimes dx^i + \frac{\sum_{i,j} x^i x^j dx^i \otimes dx^j}{1-(x^1)^2\ldots}.$$

   Is every Riemannian manifold isometrically embeddable in Euclidean space? The answer is yes, thanks to John Nash. The proof involved a new technique (the Nash-Moser inverse function theorem) that had a profound impact on analysis.

3. Products and warped products: If (M_i, g_i) are Riemannian manifolds, there is a product Riemannian metric on $M_1 \times M_2$ defined as $(g_1 \times g_2)(X_1 \oplus Y_1, X_2 \oplus Y_2) = g_1(X_1, X_2) + g_2(Y_1, Y_2)$. In terms of product coordinates, the matrix of g is block-diagonal. A warped product is as follows: Let $f : M_1 \to \mathbb{R}$ be a smooth nowhere vanishing function. Then $g = g_1 \times f^2 g_2$ is called a warped product. Several familiar metrics are warped products (at least locally) as we shall see in the HW.

4. Riemannian submersions and coverings: A smooth map $F : M \to N$ is said to be a submersion if F_* is surjective everywhere. Given a submersion, the "vertical" space V_p at $p \in M$, is the kernel of F_* at p. The horizontal space H_p is the ortho complement of V_p using g. $F : (M, g) \to (N, h)$ is said to be a Riemannian submersion, if F_* restricted to H_p is an isometry for all p. For instance, the projection map $F : M_1 \times M_2 \to M_1$ is a Riemannian submersion.
   A smooth covering map $F : M \to N$ is a smooth onto map such that every point has a neighbourhood U whose pre-image $F^{-1}(U)$ is a disjoint union of open sets each of which is diffeomorphic to U via F. In particular, F is a submersion. If $(M, g), (N, h)$ are Riemannian manifolds, the cover is said to be a Riemannian cover if it is a Riemannian submersion. Equivalently, since the vertical space is trivial, it is a local isometry (the diffeomorphisms from different sheets are isometries). Given a cover and a metric h on N, there is a unique one on M that makes it into a Riemannian cover (why?) We now construct a Riemannian metric on a torus $\mathbb{R}^n/\mathbb{Z}^n$ where the $\mathbb{Z}^n$ lattice is $\sum_i c_i v_i$ where v_i is a basis of $\mathbb{R}^n$ and c_i are integers. Indeed take the usual Euclidean metric $g = \sum_i dx^i \otimes dx^i$

on $\mathbb{R}^n$. This metric is invariant under translations by the lattice. Thus, at least on paper it seems to descend to the torus. Indeed, rigorously, note that dx^i are *globally* defined forms on the torus that trivialise its cotangent bundle. Simply declare $g = \sum_i dx^i \otimes dx^i$. It is easy to see that this defines a Riemannian covering. Thus we can attempt to generalise this construction when a Lie group G acts on a Riemannian manifold via isometries. Hopefully M/G is a manifold such that the quotient map is covering map and the Riemannian metric descends to give a Riemannian cover. Unfortunately, this expectation fails: Take $\mathbb{R}^2$ with the $\mathbb{Z}_2$ action $a \to -a$. The quotient is not even Hausdorff! (A digression: If you remove the origin, then it is Hausdorff. This raises an interesting question of what you ought to remove. In some special manifolds, the things needed to remove come from algebra. This subject is called geometric invariant theory.) One problem seems to be the presence of fixed points.
Def: We say that an action is (fixed-point) free if the isotropy group $G_x = \{g \in G | gx = x\}$ for every x is trivial.
This is not good enough for the quotient to be a manifold. For example, $\mathbb{R}$ acts on $S^1 \times S^1$ as $t.(w, z) = (\exp(2\pi i t)w, \exp(2\pi \alpha t)z)$ where α is irrational. This action is free. The orbits can be shown to be dense (maybe HW). Thus, one cannot separate the orbits and the quotient is not Hausdorff.
Basically, if you take a sequence of group elements going off to “infinity", and it the orbit of a point has a limit point, there could be a problem. Indeed, in the worst case orbits can “intersect" at “infinity" (and yet the space can be Hausdorff like the example of say $\mathbb{CP}^n$) but if there is a limit point, then such an “intersection" can happen earlier.
Thus we make a definition: G is said to act properly on M if $G \times M \to M \times M$ given by $(g, p) \to (g.p, p)$ is a proper map (this is equivalent to $G_K = \{g \in G | g.K \cap K \neq \phi\}$ is compact if K is compact), that is, the preimage of a compact set is compact. It turns out that quotients by proper actions are Hausdorff.
Here is the quotient manifold theorem: Let G act freely, smoothly and properly on M. Then M/G is a topological manifold of dim $dim(M) - dim(G)$ with a unique smooth structure such that the quotient map is a smooth submersion. If G is discrete, then under these hypotheses, the quotient map is a smooth covering map.
One can prove a Riemannian extension of the above result:
If $G \subset Isom(M, g)$, then there is a unique smooth Riemannian metric on the quotient such that the map is a Riemannian submersion.
Recall that $Aut(N)$ is the group of smooth deck transformations of a smooth cover $N \to M$. If this group is a subgroup of the isometry group, the above result implies that the quotient inherits a unique metric such that the cover is a Riemannian cover.
Examples:

(a) Consider $\mathbb{Z}$ acting on $\mathbb{R} \times \mathbb{R}$ as $n.(x, y) = (x + n, y)$. The quotient is clearly $S^1 \times \mathbb{R}$. It inherits the Euclidean metric because translations are isometries.

(b) In the example of the torus outlined above, such tori are called flat tori. Now the point is that different lattices can give rise to different metrics! (HW)

(c) S^{2n+1}/S^1 is $\mathbb{CP}^n$ (why?) and inherits a natural metric (why?) called the Fubini-Study metric.

Before we proceed further, we note the following: Given any smooth Riemannian metric and a point p, there exists a neighbourhood U and $n = dim(M)$ smooth vector fields E_i on U such that $E_i(q)$ form an orthonormal basis of T_qM for all $q \in U$.
Indeed, choose any coordinate chart around p. Perform the Gram-Schmidt procedure to convert the vector fields ∂_i to an orthonormal basis. This process involves algebraic operations and square roots of positive functions. Thus the basis is smooth.
In fact, by means of a constant linear transformation, we can assume without loss of generality that $E_i(p) = \partial_i(p)$, that is, $g_{ij} = I + O(|x|)$. As you will show in the HW, one can prove that there exist coordinates such that $g_{ij} = I + O(|x|^2)$. Such coordinates are called normal coordinates (not to be confused with a very specific choice of normal coordinates called geodesic normal coordinates that we will deal with, later). One could wonder if the second-order term can be gotten rid of. Unfortunately such is not the case. In fact, Riemann proved (more or less) that the ability to get rid of the second order term at all points is equivalent to finding coordinates where the metric is Euclidean. This second-order obstruction turns out to be related to the Riemann curvature tensor.

2.1 Induced metrics on tensor bundles

Firstly, given a Riemannian metric g on M, recalling that it is simply a smoothly varying inner product on TM, we can define a smooth metric h on a vector bundle V as an inner product on each of the fibres V_p such that it is smoothly varying, i.e., for any local trivialising sections e_i, $h(e_i, e_j)$ is a local smooth function. Alternatively, it is a smooth section of the tensor bundle $V^* \times V^*$ that is an inner product on each fibre.
Now we can define an induced Riemannian metric on T^*M (more generally for V^*) by first defining the so-called musical isomorphism $b : T_pM \to T_p^*M$ as $v^b(w) = g(v, w)$. This isomorphism is actually a smooth vector bundle isomorphism (why?) Now define $g^*(v^b, w^b) = g(v, w)$. In local coordinates, $(v^b)_i = g_{ij}v^j$ (this is also called lowering an index in physics terminology), i.e., $(v^b) = Gv$ and hence $G^{-1}(v^b) = v$. Its inverse is called $\#$ (or raising indices). Now $\langle v, w\rangle = v^T G w$ and $\langle \omega, \eta\rangle = \omega^T \tilde{G}\eta = (G^{-1}\omega)^T G G^{-1}\eta = \omega^T (G^{-1})^T \eta$ and hence $\tilde{G} = (G^{-1})^T$. We denote its local components as g^{ij}.
At this juncture, we can define the gradient of a function $f : M \to \mathbb{R}$ as a vector field: $\nabla f = (df)^\#$, i.e., $(\nabla f)^i = g^{ij}\partial_j f$. So the gradient needs a Riemannian metric for its definition (indeed, the gradient is supposed to be "normal" to the level sets).
We can raise and lower indices on tensors too. We can also define inner products among tensors. Indeed, given inner products on V and W, there is a natural "tensor" inner product on $V \otimes W$ (that is, $\langle v \otimes w, a \otimes b\rangle = \langle v, a\rangle\langle w, b\rangle$ extended linearly). Thus $\langle S, T\rangle = S^{i_1 i_2 \ldots}_{j_1 j_2 \ldots} g_{i_1 a_1} g_{i_2 a_2} \ldots g^{j_1 b_1} \ldots T^{a_1 a_2 \ldots}_{b_1 b_2 \ldots}$. For instance, for 2-forms, here is an example $\|dx \wedge dy\|^2_{Euc} = \|dx \otimes dy - dy \otimes dx\|^2 = 2$.

2.2 Volume form

Let (M, g) be an orientable Riemannian manifold. We wish to define a top-form vol_g (called the volume form) such that its integral over M must give the volume/surface area of M. To this end, if we consider a local orthonormal cobasis $\omega_1, \ldots, \omega_n$, then $\omega_1 \wedge \omega_2 \ldots \omega_n$ ought to give the infinitesimal area/volume of a square provided the basis $e_1, \ldots, e_n$ is compatible with the orientation. Suppose we choose another orthonormal oriented cobasis $\eta_1, \ldots, \eta_n$, then $\eta_i = P_i^j \omega_j$ where P is an orthogonal matrix-valued function whose determinant is $+1$ (because the orientation is compatible). Thus $\eta_1 \wedge \eta_2 \ldots \eta_n = \det(P)\omega_1 \wedge \omega_2 \ldots = \omega_1 \wedge \ldots$ (because of the transformation rule for top-forms). Thus the form $vol_g = \omega_1 \ldots$ is a smooth nowhere vanishing globally defined form that is compatible with the orientation. This form is called the volume form. In local oriented coordinates, it is $\sqrt{\det(g)} dx^1 \wedge dx^2 \ldots$. Indeed, this expression coincides with vol_g in case the coordinate vector fields are chosen to be orthonormal at a point. Moreover, if we change oriented coordinates, it changes to $\sqrt{\det(g) \det(\frac{\partial x^i}{\partial y^j})^2} \det(\partial y^j / \partial x^i) dx^1 \wedge \ldots = \sqrt{\det(g)} dx^1 \wedge dx^2 \ldots$. Thus it is a well-defined global orientation compatible nowhere vanishing form that coincides with vol_g at every point by choosing the right coordinates. Hence it is vol_g. Now we can define the integrals of functions using $\int_M f vol_g$.

3 Distance on a Riemannian manifold