1 Recap

1. Definition and existence of Riemannian metrics.

2. Constructing new ones from old ones: Conformal changes, submanifolds, products and warped products, and Riemannian submersions and covers.

2 Riemannian metrics

Riemannian submersions and coverings: Given a cover and a metric h on N, there is a unique one on M that makes it into a Riemannian cover (why?) We now construct a Riemannian metric on a torus $\mathbb{R}^n/\mathbb{Z}^n$ where the $\mathbb{Z}^n$ lattice is $\sum_i c_i v_i$ where v_i is a basis of $\mathbb{R}^n$ and c_i are integers. Indeed take the usual Euclidean metric $g = \sum_i dx^i \otimes dx^i$ on $\mathbb{R}^n$. This metric is invariant under translations by the lattice. Thus, at least on paper it seems to descend to the torus. Indeed, rigorously, note that dx^i are *globally* defined forms on the torus that trivialise its cotangent bundle. Simply declare $g = \sum_i dx^i \otimes dx^i$. It is easy to see that this defines a Riemannian covering. Thus we can attempt to generalise this construction when a Lie group G acts on a Riemannian manifold via isometries. Hopefully M/G is a manifold such that the quotient map is covering map and the Riemannian metric descends to give a Riemannian cover. Unfortunately, this expectation fails: Take $\mathbb{R}^2$ with the $\mathbb{R}^*$ action $a \to \lambda a$. The quotient is not even Hausdorff! (A digression: If you remove the origin, then it is Hausdorff. This raises an interesting question of what you ought to remove. In some special manifolds, the things needed to remove come from algebra. This subject is called geometric invariant theory.) One problem seems to be the presence of fixed points.
Def: We say that an action is (fixed-point) free if the isotropy group $G_x = \{g \in G | gx = x\}$ for every x is trivial.
This is not good enough for the quotient to be a manifold. For example, $\mathbb{R}$ acts on $S^1 \times S^1$ as $t.(w, z) = (\exp(2\pi i t)w, \exp(2\pi \alpha t)z)$ where α is irrational. This action is free. The orbits can be shown to be dense (maybe HW). Thus, one cannot separate the orbits and the quotient is not Hausdorff.
Basically, if you take a sequence of group elements going off to “infinity", and it the orbit of a point has a limit point, there could be a problem. Indeed, in the worst case orbits can “intersect" at “infinity" (and yet the space can be Hausdorff like the example of say $\mathbb{CP}^n$) but if there is a limit point, then such an “intersection" can happen earlier.
Thus we make a definition: G is said to act properly on M if $G \times M \to M \times M$ given by $(g, p) \to (g.p, p)$ is a proper map (this is equivalent to $G_K = \{g \in G | g.K \cap K \neq \phi\}$ is compact if K is compact), that is, the preimage of a compact set is compact.
It turns out that quotients by proper actions are Hausdorff.
Here is the quotient manifold theorem: Let G act freely, smoothly and properly on M. Then M/G is a topological manifold of dim $dim(M) - dim(G)$ with a unique smooth structure such that the quotient map is a smooth submersion. If G is discrete, then under these hypotheses, the quotient map is a smooth covering map.
One can prove a Riemannian extension of the above result:
If $G \subset Isom(M, g)$, then there is a unique smooth Riemannian metric on the quotient such that the map is a Riemannian submersion.

Recall that $Aut(N)$ is the group of smooth deck transformations of a smooth cover $N \to M$. If this group is a subgroup of the isometry group, the above result implies that the quotient inherits a unique metric such that the cover is a Riemannian cover.
Examples:

1. Consider $\mathbb{Z}$ acting on $\mathbb{R} \times \mathbb{R}$ as $n.(x, y) = (x + n, y)$. The quotient is clearly $S^1 \times \mathbb{R}$. It inherits the Euclidean metric because translations are isometries.

2. In the example of the torus outlined above, such tori are called flat tori. Now the point is that different lattices can give rise to different metrics! (HW)

3. S^{2n+1}/S^1 is $\mathbb{CP}^n$ (why?) and inherits a natural metric (why?) called the Fubini-Study metric.

Before we proceed further, we note the following: Given any smooth Riemannian metric and a point p, there exists a neighbourhood U and $n = dim(M)$ smooth vector fields E_i on U such that $E_i(q)$ form an orthonormal basis of T_qM for all $q \in U$.
Indeed, choose any coordinate chart around p. Perform the Gram-Schmidt procedure to convert the vector fields ∂_i to an orthonormal basis. This process involves algebraic operations and square roots of positive functions. Thus the basis is smooth.
In fact, by means of a constant linear transformation, we can assume without loss of generality that $E_i(p) = \partial_i(p)$, that is, $g_{ij} = I + O(|x|)$. As you will show in the HW, one can prove that there exist coordinates such that $g_{ij} = I + O(|x|^2)$. Such coordinates are called normal coordinates (not to be confused with a very specific choice of normal coordinates called geodesic normal coordinates that we will deal with, later). One could wonder if the second-order term can be gotten rid of. Unfortunately such is not the case. In fact, Riemann proved (more or less) that the ability to get rid of the second order term at all points is equivalent to finding coordinates where the metric is Euclidean. This second-order obstruction turns out to be related to the Riemann curvature tensor.