1 Recap

1. Riemannian submersions and the quotient manifold theorem.
2. Orthonormal frames and normal coordinates.

1.1 Induced metrics on tensor bundles

Firstly, given a Riemannian metric g on M, recalling that it is simply a smoothly varying inner product on TM, we can define a smooth metric h on a vector bundle V as an inner product on each of the fibres V_p such that it is smoothly varying, i.e., for any local trivialising sections e_i, $h(e_i, e_j)$ is a local smooth function. Alternatively, it is a smooth section of the tensor bundle $V^* \times V^*$ that is an inner product on each fibre.
Now we can define an induced Riemannian metric on T^*M (more generally for V^*) by first defining the so-called musical isomorphism $b : T_pM \to T_p^*M$ as $v^b(w) = g(v, w)$. This isomorphism is actually a smooth vector bundle isomorphism (why?) Now define $g^*(v^b, w^b) = g(v, w)$. In local coordinates, $(v^b)_i = g_{ij}v^j$ (this is also called lowering an index in physics terminology), i.e., $(v^b) = Gv$ and hence $G^{-1}(v^b) = v$. Its inverse is called $\#$ (or raising indices). Now $\langle v, w\rangle = v^T Gw$ and $\langle \omega, \eta\rangle = \omega^T \tilde{G}\eta = (G^{-1}\omega)^T GG^{-1}\eta = \omega^T (G^{-1})^T \eta$ and hence $\tilde{G} = (G^{-1})^T$. We denote its local components as g^{ij}.
At this juncture, we can define the gradient of a function $f : M \to \mathbb{R}$ as a vector field: $\nabla f = (df)^\#$, i.e., $(\nabla f)^i = g^{ij}\partial_j f$. So the gradient needs a Riemannian metric for its definition (indeed, the gradient is supposed to be "normal" to the level sets).
We can raise and lower indices on tensors too. We can also define inner products among tensors. Indeed, given inner products on V and W, there is a natural "tensor" inner product on $V \otimes W$ (that is, $\langle v \otimes w, a \otimes b\rangle = \langle v, a\rangle\langle w, b\rangle$ extended linearly). Thus $\langle S, T\rangle = S^{i_1 i_2 \ldots}_{j_1 j_2 \ldots} g_{i_1 a_1} g_{i_2 a_2} \ldots g^{j_1 b_1} \ldots T^{a_1 a_2 \ldots}_{b_1 b_2 \ldots}$. For instance, for 2-forms, here is an example $\|dx \wedge dy\|^2_{Euc} = \|dx \otimes dy - dy \otimes dx\|^2 = 2$.

1.2 Volume form

Let (M, g) be an orientable Riemannian manifold. We wish to define a top-form vol_g (called the volume form) such that its integral over M must give the volume/surface area of M. To this end, if we consider a local orthonormal cobasis $\omega_1, \ldots, \omega_n$, then $\omega_1 \wedge \omega_2 \ldots \omega_n$ ought to give the infinitesimal area/volume of a square provided the basis $e_1, \ldots, e_n$ is compatible with the orientation. Suppose we choose another orthonormal oriented cobasis $\eta_1, \ldots, \eta_n$, then $\eta_i = P_i^j \omega_j$ where P is an orthogonal matrix-valued function whose determinant is $+1$ (because the orientation is compatible). Thus $\eta_1 \wedge \eta_2 \ldots \eta_n = \det(P)\omega_1 \wedge \omega_2 \ldots = \omega_1 \wedge \ldots$ (because of the transformation rule for top-forms). Thus the form $vol_g = \omega_1 \ldots$ is a smooth nowhere vanishing globally defined form that is compatible with the orientation. This form is called the volume form. In local oriented coordinates, it is $\sqrt{\det(g)}dx^1 \wedge dx^2 \ldots$. Indeed, this expression coincides with vol_g in case the coordinate vector fields are chosen to be orthonormal at a point. Moreover, if we change oriented coordinates, it changes to $\sqrt{\det(g)\det(\frac{\partial x^i}{\partial y^j})^2} \det(\partial y^j/\partial x^i) dx^1 \wedge \ldots = \sqrt{\det(g)}dx^1 \wedge dx^2 \ldots$. Thus it is a well-defined global orientation compatible nowhere vanishing form that coincides with vol_g at every point by choosing the right coordinates. Hence it is vol_g. Now we can define the integrals of functions using $\int_M f vol_g$.

2 Distance on a Riemannian manifold

To make a Riemannian manifold-with-boundary (M, g) into a metric space, one considers $d_g(p, q)$ to be the infimum of the lengths $l(\gamma) = \int_a^b \|\gamma'\| dt$ over all piecewise smooth regular (that is, $\gamma'(t) \neq 0$) paths $\gamma : [a, b] \to M$. Such paths are called admissible paths (and their images are typically called admissible curves). A reparametrisation is a homeomorphism between intervals that is a diffeomorphism on the subintervals where γ is smooth and regular. It is easy to see that the length is additive (timewise), reparametrisation invariant, and isometry invariant. It is also easy to see that every admissible curve has a unique forward reparametrisation by arc-length.

For the distance function to make sense, we need non-emptyness of the set where the infimum is taken.

Prop: If M is a connected manifold-with-boundary, any two points can be connected by an admissible curve.

Proof. Given p, let S be the set of all points in M that can be connected by an admissible curve from p. It is non-empty (p is in it) and open: Suppose p is connected to q, then q can be connected to nearby points lying in a coordinate chart. It is also closed: Indeed, if $q_n \to q$, then q can be connected to q_n for large n such that it lies in a coordinate neighbourhood of q. □