1 Recap

1. Induced metrics on tensor bundles. Gradient.
2. Volume form.
3. Admissible curves, distance function, and the fact that it is well-defined on connected manifolds-with-boundary.

2 Distance on a Riemannian manifold

It is easy to see that this distance function is an isometry-invariant.
Now we prove an important theorem: *Let (M, g) be a connected Riemannian manifold-with-boundary. The distance function $d_g(p, q)$ makes M into a metric space with the same topology.*

Proof. d_g is obviously symmetric (just reverse the direction of the paths) and $d_g \geq 0$. Consider admissible paths γ_1, γ_2 connecting (p, q) and (q, r) respectively such that $d(p, q) \leq l(\gamma_1) \leq d(p, q) + \frac{\epsilon}{2}$ and likewise for $d(q, r)$. Now γ_1 concatenated with γ_2 gives an admissible path γ from p to r and $d(p, q) + d(q, r) \leq l(\gamma) \leq d(p, q) + d(q, r) + \epsilon$. Since $l(\gamma) \geq d(p, r)$ we see that $d(p, r) \leq d(p, q) + d(q, r) + \epsilon \ \forall \epsilon > 0$ and hence the triangle inequality holds. If $d(p, q) = 0$, then we need to prove that $p = q$: If p and q are distinct, there are two non-intersecting coordinate balls (of Euclidean radius 2) with centred around p and q. Now the Riemannian metric is comparable to the Euclidean metric in these balls in that $\frac{1}{C} g_{Euc} \leq g \leq C g_{Euc}$. Thus, $\frac{1}{C} l_{Euc}(\gamma) \leq l(\gamma) \leq C l_{Euc}(\gamma)$. Now $l_{Euc}(\gamma)$ is the length of the straight line joining the two points (why?) and hence there is a minimum distance separating the two points.
Consider a basis of coordinate balls for M (of Euclidean radius at most 1) centred at each point $p \in M$ such that they are contained in coordinate 2-balls. By the aforementioned calculation, each ball contains a metric ball and any metric ball of radius at most $\frac{1}{C}$ contains a coordinate ball. Hence the basis of metric balls gives rise to the same topology as the Euclidean ones. □

3 The energy functional and geodesics

Naively one can try to minimise $l(\gamma)$ among admissible curves using calculus of variations. That is, suppose $0 \in [c, d]$ and $\gamma(t, s) : [a, b] \times [c, d]$ is a variation of admissible curves, i.e., it is continuous, on $[a_i, a_{i+1}] \times [c, d]$ it is smooth and for each fixed s, $\gamma(t, s)$ is an admissible curve. If $l(\gamma)$ achieves a minimum at some admissible curve γ_0, then for any admissible variation (such that $\gamma(t, 0) = \gamma_0$ and it fixes the endpoints), the derivative of $l(\gamma(., s))$ at $s = 0$ ought to be zero if $l(\gamma(., s))$ is differentiable. That will gives us an equation for such curves (which we can then hope, have solutions, and are genuine minimisers). However, l is painful because of the square root. A substitute is to consider $E(\gamma) = \int_a^b \|\gamma'\|^2 dt$. Then suppose we differentiate a *smooth* variation assuming we have a *smooth* minimiser,

$$\frac{dE}{ds} = \int_a^b \frac{dg_{ij}}{ds}(\gamma')^i(\gamma')^j dt + \int_a^b 2g_{ij}\frac{d(\gamma')^i}{ds}(\gamma')^j$$

$$= \int_a^b g_{ij,k}\gamma^k_{,s}(\gamma')^i(\gamma')^j dt - \int_a^b 2g_{kj,i}(\gamma^i)'\frac{d\gamma^k}{ds}(\gamma')^j - \int_a^b 2g_{ij}\frac{d\gamma^i}{ds}(\gamma^j)''$$

$$= 2\int_a^b \left(-g_{kl}\Gamma^l_{ij}(\gamma')^i(\gamma')^j - g_{kj}(\gamma^j)''\right)\gamma^k_{,s}dt, \tag{1}$$

where $2g_{kl}\Gamma^l_{ij} = g_{kj,i} - g_{ij,k} + g_{ki,j}$. (These Γ's are called Christoffel symbols.) Moreover, $V^k = \gamma^k_{,s}$ is a vector field along the curve (at $s = 0$) that is zero at the endpoints. It is smooth at smooth points (if we take non-smooth variations along non-smooth minimisers), i.e., can be extended locally to a smooth vector field (why? *HW*). Likewise, γ' is also a vector field that is smooth at smooth points. Now we see that $2\int_a^b f_k\gamma^k_{,s}dt = 0$ for all such $\gamma^k_{,s}$. This implies that f_k is 0 at all smooth points. Indeed, in a coordinate neighbourhood of one, simply choose the vector field $V = \sum_k f_k\partial_k\rho$ where ρ is a smooth bump function. The variation is $\gamma(t,s) = \gamma(t)$ outside the neighbourhood and $F(\gamma(t), s)$ where F is the flow of V inside. Thus f_k is zero in that neighbourhood. We shall extend this argument to the more general piecewise smooth case later.

This gives rise to the geodesic equation: $(\gamma^i)'' + \Gamma^i_{jk}(\gamma^j)'(\gamma^k)' = 0$. Note that the Christoffel symbols are not components of a $(1,2)$ tensor because they do not transform correctly. Solutions of this equation are called geodesics. It is not immediately obvious that this equation makes sense globally (*HW*). One can now try to study the relationship between these geodesics and length-minimising curves. However, one might raise a valid objection: Why not look at the L^4 energy or the L^6 energy or some such thing? We need more justification to claim that this is the right set of curves to study.

4 Connections and the Levi-Civita connection

Recall that on surfaces, one of the ways we looked at curvature was to parallel transport a vector along a small curve. The difference was related to the curvature. Note that this concept of parallel transport also has bearing on length minimising curves. How do you minimise the distance you travel? You literally follow your nose! That is, you parallel transport your velocity! How does one model this phenomenon mathematically? We can try the following in the case of say submanifolds of Euclidean space: A smooth vector field V over a smooth curve γ is parallel to itself if $\frac{DV}{dt} = \frac{dV}{dt} - \pi(\frac{dV}{dt}) = 0$ where π is the orthogonal projection to the normal vector space. (It is not immediately obvious that this definition does not depend on how you extend V smoothly locally but that's an exercise.) On a general manifold, we do not have this luxury.