1 Recap

1. Distance function is a metric with the same topology. (We did not prove that the distance function on Euclidean space is the usual one. Here is the proof: Approximate by a piecewise linear curve and use the triangle inequality.)

2. Geodesic equation via variation of the Energy functional for smooth critical points.

3. Motivation for parallel transport through derivatives.

2 Connections and the Levi-Civita connection

On a general manifold, we can still try to define $\frac{DV}{dt}$, or more generally, the directional derivative of V along some vector field X (in the case of a curve, it is $X = \gamma'$) $\nabla_X V$ using abstract properties. More generally, we can try to differentiate *sections* s of vector bundles E with respect to a direction field X, $\nabla_X s$. To this end, we define:
A connection $\nabla_X s$ is a map taking smooth vector fields X and smooth sections s of a vector bundle E, and producing smooth sections of E (this is crucial - indeed, $\frac{DV}{dt}$, the acceleration, is a smooth tangent field on the submanifold) such that

1. $\nabla_{f_1 X + f_2 Y} s = f_1 \nabla_X s + f_2 \nabla_Y s$ for smooth functions f_1, f_2 and smooth vector fields X, Y. This property indicates that the directional derivative at a point genuinely depends on the direction at that point alone. (and is not a differential operator that needs the behaviour of the vector field at other points).

2. If a_1, a_2 are constants, then $\nabla_X(a_1 s_1 + a_2 s_2) = a_1 \nabla_X s_1 + a_2 \nabla_X s_2$.

3. $\nabla_X(fs) = X(f)s + f\nabla_X s$ (the Leibniz rule -indicating that indeed ∇ depends on the first derivative of the section).

We can now easily prove (how?) that $\nabla_X s$ depends only on the local behaviour of s and only on the value of X at the point.

The next order of business is to see what a connection looks like locally. Suppose we choose a trivialising collection of local smooth sections $e_1, \ldots, e_r$ for the vector bundle, then

$$\nabla_X(s^i e_i) = X(s^i)e_i + s^i \nabla_X e_i. \tag{1}$$

Now if we choose coordinates for the manifold, then

$$\begin{aligned}\nabla_X(s^i e_i) &= X^\mu \partial_\mu s^i e_i + s^i X^\mu \nabla_{\partial_\mu} e_i \\ &= X^\mu \left(\partial_\mu s^i + A^i_{\mu\ j} s^j\right) e_i,\end{aligned} \tag{2}$$

where $\nabla_{\partial_\mu} e_j = A^i_{\mu\ j} e_i$. These $A's$ are called connection coefficients. (It turns out that the Christoffel symbols are connection coefficients for a (unique) connection on TM called the Levi-Civita connection.) There is a slightly different way to look at connections. Note that since $\nabla_X s(p)$ depends purely on $X(p)$, the map $X(p) \to \nabla_X s(p)$ is a smooth vector bundle morphism from TM to E (why? This concept is sometimes called "tensoriality in X"). In other words, ∇s can be thought of as a section of $E \otimes T^*M$ (why?) and is sometimes called a "vector-valued 1-form". Thus $\nabla s = (something)^i_\mu dx^\mu \otimes e_i$. The $(something)^i_\mu$ is precisely $\partial_\mu s^i + A^i_{\mu\ j} s^j$. It can also be written locally as $\nabla s = (ds^i + A^i_j s^j) e_i$ or even more simply, $ds + As$ where A is a matrix of 1-forms called the connection matrix. Here is an important proposition:

Connections exist on every vector bundle E over a manifold M.

Proof: Cover M with trivialising open sets U_α and let ρ_α be a partition-of-unity subordinate to them. Define ∇s as $\nabla s = \sum_\alpha \rho_\alpha ds^i_\alpha \otimes e_{i,\alpha}$. This sum is locally finite and hence well-defined. It is easy to check that the axioms of a connection are met. $\square$