NOTES FOR 11 NOV (TUESDAY)

1. Recap

(1) Trace theorem.
(2) Gagliardo-Sobolev-Nirenberg inequality. The C^0 part of the Morrey inequality.

1.1. Sobolev embedding.

Theorem 1.1. *For all $u \in C^1(\mathbb{R}^n)$, $\|u\|_{C^{0,\gamma}} \leq C\|u\|_{W^{1,p}}$ where $\gamma = 1 - \frac{n}{p}$.*

Proof. For a Hölder bound, we want to estimate $|u(x) - u(y)|$ in terms of $x - y$. Now let $r = \|x - y\|$ and $W = B(x, r) \cap B(y, r)$. Then

$$
\begin{aligned}
|u(x) - u(y)| &\leq \fint_W |u(x) - u(z)| dz + \fint_W |u(z) - u(y)| dz \\
&\leq Cr^{1-n/p}\|Du\|_{L^p}. \quad (1.1)
\end{aligned}
$$

This shows the desired bound. □

Now we have one more Sobolev embedding: Suppose U is bounded open with C^1 boundary and $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u agrees a.e. with a $C^{0,\gamma}(U)$ function v with $\|v\|_{C^{0,\gamma}} \leq C\|u\|_{W^{1,p}(U)}$.

Proof. We prove $p < \infty$ (and $p = \infty$ is an exercise). As before, we extend u to Eu. By convolution, we approximate by smooth functions u_m with compact support on $\mathbb{R}^n$. These functions converge (thanks to the above inequality) to a $C^{0,\gamma}(U)$ function v. □

Inductively we get general Sobolev inequalities for $W^{k,p}$. The key point is that if you are in $W^{k,2}$ for all k, then you are smooth.

1.2. Solving the Poisson equation weakly.

We shall now prove that there exists an $H_0^1(U)$ function u satisfying $\Delta u = f$ in a weak sense (where f is smooth on $\bar{U}$). Indeed, consider $B[u, v] = \int \nabla u . \nabla v$ on $H_0^1(U)$. By the Poincaré inequality, this is an inner product and equivalent to the Sobolev norm. Consider the linear functional $F(u) = -\int uf$. It is bounded linear and hence by Riesz representation there exists a unique $u \in H_0^1(U)$ such that $B[u, v] = F(v)$ for all $v \in H_0^1(U)$. In particular, it is a distributional solution. We now need to prove that u is smooth.

1.3. Difference quotients.

Suppose $u : U \to \mathbb{R}$ is in L^1_{loc} and $D_i^h u = \frac{u(x+he_i)-u(x)}{h}$. We have a similar theorem (as before about difference quotients). Let $1 < p < \infty$, $V \subset\subset U$, and $0 < h < \frac{1}{2}d(V, \partial U)$.

Theorem 1.2. (1) *Suppose $u \in W^{1,p}(U)$. Then $\|D^h u\|_{L^p}(V) \leq C\|Du\|_{L^p}(U)$ for all $0 < h < \frac{1}{2}d(V, \partial U)$.*
(2) *Suppose $u \in L^p(V)$ and $\|D^h u\|_{L^p(V)} \leq C$. Then $u \in W^{1,p}(V)$ and $\|Du\|_{L^p(V)} \leq C$.*

Proof. (1) By approximation, assume WLog that u is smooth. Then using FTC and Fubini $\|D^h u\|_{L^p}(V)^p \leq C \int_0^1 \int_V \|Du(x + the_i)\|^p \leq C\|Du\|_{L^p}^p$.

(2) Suppose ϕ is a test function. Then we can prove the "integration-by-parts" formula $\int_V uD_i^h\phi = -\int D_i^{-h}u\phi$. The given estimate and Banach-Alagolu implies that passing to a subsequence, $D_i^{-h_k}u$ goes weakly to v_i in $L^p(V)$. We can easily show that $v_i = u_{x_i}$ in the weak sense and hence we are done.

□

This proof applies even to the tangential derivatives when V is a half-ball in the upper half-space.

1.4. **Regularity - Interior.** We shall prove that all the Sobolev norms (on a compact set V) are bounded and hence u is smooth in the interior. Suppose u is already smooth up to the boundary (and zero on the boundary). Let's prove estimates then (the non-smooth case will involve replacing derivatives by difference quotients). Let $\zeta \equiv 1$ on V and compactly supported in U.

$$\Delta u = f \Rightarrow \int \sum_{i,j} \zeta^2 u_{ii}u_{jj} = \int \zeta^2 f^2$$

$$\Rightarrow -\int ((\zeta^2)_j u_{ii}u_j + \zeta^2 u_{iij}u_j) = \int \zeta^2 f^2$$

$$\Rightarrow \int (-(\zeta^2)_j u_{ii}u_j + (\zeta^2)_i u_{ij}u_j + \zeta^2 u_{ij}^2) = \int \zeta^2 f^2. \tag{1.2}$$

Using Cauchy-Schwarz we can conclude that $\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}) \leq C\|f\|_{L^2(U)}$ because $u \in H_0^1(U)$. Differentiating the equation and using this estimate over and over, we get estimates for Sobolev norms. However, the catch is that we cannot literally say that $\Delta u = f$. We can only say it in a weak sense. So the above calculation needs to be modified for the non-smooth case. We first note the following useful "calculus" for v being compactly supported.

$$\int_U vD_k^{-h}w = -\int wD_k^h v$$

$$D_k^h(vw) = v^h D_k^h w + wD_k^h v. \tag{1.3}$$

Note that $B[u,v] = \int \nabla u . \nabla v = -\int fv$ for all $v \in H_0^1(U)$. Motivated by the calculation for smooth ones and by the presence of $-h$ in the calculus above, we try $v = -D_k^{-h}(\zeta^2 D_k^h u)$. Therefore

$$\int D_k^h(\nabla u).\nabla(\zeta^2 D_k^h u) = \int fD_k^{-h}(\zeta^2 D_k^h u)$$

$$\Rightarrow \int_V D_k^h(\nabla u)^2 \leq C\|u\|_{H^1(U)}^2 + C\|f\|_{L^2(U)}^2 \leq C\|f\|_{L^2(U)}^2. \tag{1.4}$$

We can prove higher regularity inductively. Indeed, we can prove that $\Delta D^\alpha u = D^\alpha f$ is satisfied in the weak sense. Of course $D^\alpha u$ does NOT have trace zero on the boundary UNLESS $\alpha_n = 0$. (Why is the latter true?) However, since we are proving interior regularity, the previous estimates continue to work. Thus $D^\alpha u$ is in $H^2(V)$ and so on. Therefore, u is smooth in the interior.

1.5. **Regularity - Boundary.** Regularity is again a local property (near the boundary). So we want to change coordinates via a local diffeomorphism $x \to y = (x', x^n - g(x'))$ to flatten the boundary (the Jacobian of this map is 1). This changes the PDE. Indeed, $B[u,v] = \sum_i \int u_i v_i dx = \int u'_\alpha \frac{\partial y^\alpha}{\partial x^i} v'_\beta \frac{\partial y^\beta}{\partial x^i} dy$ and $\int vf dx = \int v' f' dy$. Thus we can instead study the new elliptic PDE $(a^{ij}u_j)_i = f$ where $a^{ij} = \sum_\alpha \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\alpha}$ on a semiball of radius 1 centred at the origin in the upper half-space. As before, we can multiply by ζ with the difference being that ζ is allowed to be supported on the upper half-space, i.e., it is zero on the curved part of the semiball. Roughly speaking we multiply by u_{ij} where i, j are in the horizontal directions. As for u_{nn}, we can solve for it using the PDE itself and hence get estimates upto the boundary.

Let $1 \leq k \leq n-1$, $1/100 > h > 0$, and $v = -D_k^{-h}(\zeta^2 D_k^h u)$ where $\zeta \equiv 1$ on $B(0, \frac{1}{2})$ and supported on $B(0,1)$. Note that $v \in H_0^1(U)$. Thus $B'[u,v] = \int fv$. We estimate as before to conclude that $u_k \in H^1(V)$, and $\sum_{i+j<2n} \|u_{ij}\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$. Using the PDE itself, we see that $\|u_{nn}\|_{L^2(V)}$ also satisfies a similar bound. Hence we have shown that for $\Delta u = f$, $u \in H^2(U)$ and $\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)})$. (In fact the same kind of estimate holds even if add first-order terms to the equation.) To prove higher-order regularity, again we change our coordinates. Then take $\tilde{u} = D^\alpha u$ where $\alpha_n = 0$. Now we can show that $\tilde{u}$ is a solution of a new PDE obtained by differentiating the old one. That will allow use to show that $\tilde{u} \in H^2(V)$. By solving using the PDE itself, we can actually upgrade the regularity to H^m for all m and hence u is smooth up to the boundary.