NOTES FOR 11 SEPT (THURSDAY)

1. Recap

(1) Curvature of Riemannian manifolds.
(2) Stokes theorem and divergence theorem.

2. Divergence, Stokes' theorem, and Laplacians

Does such an operator $* : \Gamma(\Omega^k(M)) \to \Gamma(\Omega^{m-k}(M))$ exist ? Is it linear ? Yes to both. Suppose $\omega_1, \omega_2, \ldots, \omega_m$ form an orthonormal frame on an open set U, i.e., $\omega_1(p), \omega_2(p), \ldots, \omega_m(p)$ form an orthonormal basis of $T_p^* M$ for all $p \in U$. Then, $*(\omega_{i_1} \wedge \omega_{i_2} \ldots \omega_{i_k}) = (-1)^{sgn(I)} \omega_{i_{k+1}} \wedge \omega_{i_{k+2}} \ldots \wedge \omega_{i_m}$ where $sgn(I)$ is the sign of the permutation taking $(1, 2, \ldots, m)$ to $(i_1, i_2, \ldots, i_m)$. Then extend $*$ linearly to all forms. We will see why it is well-defined later on. Here are some examples :

(1) Suppose $(M, g) = \mathbb{R}^2, g_{Euc}$ oriented in the usual way, then $*1 = dx \wedge dy$. Also, $*dx = dy$ and $*dy = -dx$. Finally, $*(dx \wedge dy) = 1$.
(2) If $M = \mathbb{R}^3$ (with the Euclidean metric) oriented in the usual way, then $*1 = dx \wedge dy \wedge dz$, $*dx = dy \wedge dz$, $*dy = dz \wedge dx$, $*dz = dx \wedge dy$.
If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, form the dual 1-forms $v = v_1 dx + v_2 dy + v_3 dz$ and likewise for w. Then $v \wedge w$ is a 2-form given by $v \wedge w = (v_1 w_2 - v_2 w_1) dx \wedge dy + \ldots$. The Hodge star acting on this gives a 1-form $*(v \wedge w) = (v_1 w_2 - v_2 w_1) dz + \ldots$ whose dual is $(v_2 w_3 - v_3 w_2, v_3 w_1 - w_3 v_1, v_1 w_2 - w_1 v_2)$ which are the components of $\vec{v} \times \vec{w}$. Since the cross product depends on the choice of orientation, it is called a "pseudovector".

This $*$ operator (the so-called Hodge star) has the following properties :

(1) Suppose α, β are elements of $\Omega_p^k(M)$ $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g vol_g = \beta \wedge *\alpha$, i.e., it does satisfy the definition.
(2) $*$ is well-defined, i.e., it does not depend on the choice of orthonormal basis.
(3) If you change the metric from g to $\tilde{g} = cg$ where $c > 0$ is a constant, then $*_{\tilde{g}} \omega = \sqrt{c}^{2k-m} *_g \omega$
(4) If you change the orientation, $* \to -*$.
(5) $* * \eta = (-1)^{k(m-k)} \eta$.
(6) $\langle *\alpha, *\eta \rangle = \langle \alpha, \eta \rangle$.

Proof. (1) Suppose we choose the orthonormal frame ω_i. Suppose $\beta = \beta_I \omega^{i_1} \wedge \omega_{i_2} \ldots$ where the summation is over increasing indices $i_1 < i_2 < \ldots$, we see that $*\beta = \beta_I (-1)^{sgn(I)} \omega^{i_{k+1}} \wedge \omega^{i_{k+2}} \ldots$. Thus,

$$\begin{aligned} \alpha \wedge *\beta &= \alpha_J \beta_I (-1)^{sgn(I)} \omega^{j_1} \wedge \omega^{j_2} \ldots \omega^{j_k} \wedge \omega^{i_{k+1}} \wedge \ldots \\ &= \alpha_I \beta_I (-1)^{sgn(I)} (-1)^{sgn(I)} \omega_1 \wedge \omega_2 \ldots = \alpha_I \beta_I vol_g = \langle \alpha, \beta \rangle vol_g \end{aligned} \quad (2.1)$$

Note that this property does not depend on how we defined $*$ (i.e., we did not use the fact that $*$ is well-defined)

(2) The above property $\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol_g$ *defines* $*$ uniquely because, if $*_1, *_2$ satisfy this property, then $\alpha \wedge (*_1 - *_2)\beta = 0$ for all α, β. However, $(a, b) \to a \wedge b$ is a non-degenerate pairing (Why? because $(a, *_1 a) \to a \wedge *_1 a = |a|^2 vol_g \geq 0$). Hence $*_1 \beta = *_2 \beta \ \forall \ \beta$.

(3) Suppose $\omega_1, \ldots, \omega_m$ is an orthonormal frame for g, then $\frac{\omega_i}{\sqrt{c}}$ is one for $\tilde{g}$. From this the result follows trivially.
(4) Obvious.
(5)

$$**(\eta) = \eta_I **(\omega^I) = \eta_I *((-1)^{sgn(I)}\omega^{I^c}) = \eta_I(-1)^{sgn(I)}(-1)^{sgn(I^c)}\omega^I = (-1)^{k(m-k)}\eta \tag{2.2}$$

(6) Suppose η is a k-form and α an $m-k$ form.

$$\begin{aligned}\langle *\alpha, *\eta\rangle vol &= *\alpha \wedge ** \eta = (-1)^{k(m-k)} * \alpha \wedge \eta \\ &= (-1)^{k(m-k)}(-1)^{k(m-k)}\eta \wedge *\alpha = \langle \eta, \alpha\rangle vol = \langle \alpha, \eta\rangle vol\end{aligned} \tag{2.3}$$

□

Now we define an operator analogous of the curl $\nabla \times \vec{F}$:

Definition 2.1. Let α be a smooth k-form. Then $d^\dagger\alpha = (-1)^{m(k+1)+1} * d * \alpha$. Thus $d^\dagger\alpha$ is a smooth $k-1$-form depending on the first derivative of α (it is a first order differential operator).

Definition 2.2. Let α be a smooth k-form. Then $d^\dagger\alpha = (-1)^{m(k+1)+1} * d * \alpha$. Thus $d^\dagger\alpha$ is a smooth $k-1$-form depending on the first derivative of α (it is a first order differential operator).

The "codifferential" satisfies the following properties :
(1) $d^\dagger f = 0$ where f is a smooth function.
(2) $d^\dagger \circ d^\dagger = 0$.
(3) $(d\alpha, \beta) = \int_M \langle d\alpha, \beta\rangle vol_g = \int_M \langle \alpha, d^\dagger\beta\rangle vol_g = (\alpha, d^\dagger\beta)$. Thus, $d^\dagger$ is formally speaking, the adjoint of d.
(4) If X is a vector field and ω_X is the dual 1-form, then $d^\dagger\omega_X = -div(X)$. Hence, $d^\dagger df = -\Delta f$.

Proof. (1) Obvious because f is a 0-form.
(2) $d^\dagger \circ d^\dagger = \pm * d *^2 d* = \pm * d \circ d* = 0$
(3) Suppose β is a k-form and α a $k-1$ form.

$$\begin{aligned}(\alpha, d^\dagger\beta) &= \int_M \alpha \wedge (-1)^{m(k+1)+1} **d * \beta = \int_M \alpha \wedge (-1)^{m(k+1)+1+(m-k)k} d * \beta \\ &= \int_M (-1)^k(-1)^k(d(\alpha \wedge *\beta) - d\alpha \wedge *\beta) = \int_M d\alpha \wedge *\beta = (d\alpha, \beta)\end{aligned} \tag{2.4}$$

(4) Note that

$$\begin{aligned}(d^\dagger\omega_X, f) &= (\omega_X, df) = \int_M g^{ij}(\omega_X)_i \frac{\partial f}{\partial x^j} vol \\ &= \int_M g^{ij} g_{ik} X^k \frac{\partial f}{\partial x^j} vol = (X, \nabla f) = -(div(X), f) \\ &\Rightarrow (d^\dagger\omega_X + div(X), f) = 0 \ \forall \ f \in C^\infty(M)\end{aligned} \tag{2.5}$$

The last equality implies the result because we can choose f to be a mollifier supported inside a coordinate chart and take limits. □

The last equality motivates us to make the following definition :

Definition 2.3. Suppose α is a smooth k-form on a compact oriented Riemannian manifold (M, g). Define the second order linear partial differential operator (the Hodge Laplacian or the Laplace-Beltrami operator) as the k-form $\Delta_d \omega = (dd^\dagger + d^\dagger d)\omega$.

Let us calculate this on $\mathbb{R}^m$ with the Euclidean metric and the usual orientation. (Remember that this Laplacian depends on the choice of a metric and an orientation.) Let $\eta = \eta_I dx^I$ be a k-form (where the sum is over all indices, whether increasing or not).

$$d\eta = \frac{\partial \eta_I}{\partial x^j} dx^j \wedge dx^I$$

$$d^\dagger \eta = (-1)^{m(k+1)+1} * d * \eta = (-1)^{m(k+1)+1} * d(\eta_I (-1)^{sgn(I,I^c)} dx^{I^c})$$

$$= (-1)^{m(k+1)+1+sgn(I,I^c)} * \frac{\partial \eta_I}{\partial x^j} dx^j \wedge dx^{I^c} = (-1)^{m(k+1)+1+sgn(I,I^c)} \frac{\partial \eta_I}{\partial x^j} (-1)^{sgn(j,I^c,i_1,\ldots)} dx^{i_1} \ldots dx^{i_{a_j(I)-1}} \wedge \hat{dx^j} \ldots$$

$$= (-1)^{m(k+1)+m-k+a_j(I)+k(m-k)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j} \ldots = (-1)^{a_j(I)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j} \ldots$$

$$\Delta_d \eta = (dd^\dagger + d^\dagger d)\eta = d((-1)^{a_j(I)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j}) + d^\dagger \frac{\partial \eta_I}{\partial x^l} dx^l \wedge dx^I$$

$$= \sum_{I,l,j \in (i_1,\ldots,i_k)} (-1)^{a_j(I)} \frac{\partial^2 \eta_I}{\partial x^l \partial x^j} dx^l \wedge dx^{i_1} \ldots \wedge \hat{dx^j} \ldots + \sum_{I,l \in I^c, j \in (l,i_1,\ldots,i_k)} (-1)^{a_j(l,I)} \frac{\partial^2 \eta_I}{\partial x^l \partial x^j} dx^l \wedge dx^{i_1} \ldots \hat{dx^j} \ldots \ldots$$

$$= -\sum_{I,k} \frac{\partial^2 \eta_I}{\partial (x^k)^2} dx^I = -(\Delta \eta_I) dx^I$$

So, in particular, in Euclidean space, if we compute the principal symbol of the Hodge Laplacian, i.e., we replace the highest order derivatives by a vector $\vec{\zeta}$, we get $\sigma_{\Delta_d}(\vec{\zeta}) = -\begin{bmatrix} |\zeta|^2 & 0 & \ldots \\ 0 & |\zeta|^2 & \ldots \\ \vdots & \ddots & \ldots \end{bmatrix}$.
Hence this operator is elliptic with constant coefficients. This holds true even for the flat torus.

Before we proceed further with the analysis of the PDE $\Delta_d \eta = \alpha$, we define a general notion of a Laplacian (the so called Bochner Laplacian or the Rough Laplacian). To do so, suppose (E, ∇, h) is a vector bundle on a compact oriented Riemannian manifold (M, g) with a metric (h) compatible connection ∇. Then we identify the formal adjoint $\nabla^\dagger : \Gamma(T^*M \otimes E) \to \Gamma(E)$ of the connection $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ defined by the property

$$(\nabla^\dagger \alpha, \beta) = \int_M \langle \nabla^\dagger \alpha, \beta \rangle_h vol_g = \int_M \langle \alpha, \nabla \beta \rangle_{g^* \otimes h} vol_g = (\alpha, \nabla \beta) \tag{2.6}$$

We need to prove that such an operator is actually a differential operator by finding a formula for it. (Such an operator is unique - Why ?) Suppose we choose an orthonormal normal trivialisation e_i for (E, ∇, h) and normal coordinates x^μ for g at p, then $A(p) = 0, h(p) = Id, g = Id + O(x^2) = g^*$. Let $\alpha = \alpha^i_\mu dx^\mu \otimes e_i$, $\beta = \beta^j e_j$. Then

$$\langle \alpha, \nabla \beta \rangle_{g^* \otimes h}(p) = \sum_{\mu,i} \alpha^i_\mu(p) \frac{\partial \beta^i}{\partial x^\mu}(p) = \sum_{\mu,i} \frac{\partial \alpha^i_\mu \beta^i}{\partial x^\mu}(p) - \frac{\partial \alpha^i_\mu}{\partial x^\mu}(p) \beta^i(p)$$

$$= div(\langle \alpha, \beta \rangle^\sharp)(p) - \frac{\partial \alpha^i_\mu}{\partial x^\mu}(p) \beta^i(p)$$

Now the expression $-\frac{\partial \alpha^i_\mu}{\partial x^\mu}(p)\beta^i(p)$ can be written as $-\langle tr(\nabla \alpha), \beta \rangle_h(p)$ which is a globally defined quantity. By the divergence theorem, $\nabla^\dagger \alpha = -tr(\nabla \alpha)$. So finally,

Definition 2.4. Suppose (M, g) is a compact oriented Riemannian manifold (without boundary as usual) and (E, ∇, h) is a vector bundle with a metric h and a metric-compatible connection ∇. The Bochner Laplacian (sometimes called the Rough Laplacian) is defined as $\nabla^{\dagger}\nabla : \Gamma(E) \to \Gamma(E)$ where $\nabla^{\dagger}\alpha = -tr(\nabla\alpha)$.

Suppose we take $E = \Omega^k(M)$, then potentially, we have two Laplacians, Δ_d and $\nabla^*\nabla$. It turns out that

$$\Delta_d \eta = \nabla^* \nabla \eta + Curvature(\eta) \tag{2.7}$$

where the last term is something that depends linearly on η with coefficients depending on the Riemann tensor. This sort of an identity relating two different Laplacians is called a Bochner-Weitzenböck identity. So, taking inner product with η and integrating,

$$(d\eta, d\eta) + (d^{\dagger}\eta, d^{\dagger}\eta) = (\nabla\eta, \nabla\eta) + (\eta, Curvature(\eta)) \geq (\eta, Curvature(\eta)) \tag{2.8}$$

So if $\Delta_d \eta = 0$, i.e., η is Harmonic, and the curvature term is positive, we have a contradiction unless $\eta = 0$. This sort of a conclusion turns out to be useful for topology. This method is called the Bochner technique for proving non-existence of non-trivial Harmonic objects.