NOTES FOR 14 AUG (THURSDAY)

1. Recap

(1) Proved some properties of mollifiers (Evans' appendix).
(2) Defined the Sobolev norm and the Sobolev space. Proved that the Sobolev space is a Hilbert space and that smooth functions are dense in it.
(3) Proved the Sobolev embedding theorem. Defined compact operators between Banach spaces.

2. Weak solutions and Sobolev spaces

Sobolev spaces give nice examples of compact operators. The point is that it in the study of PDE, we want to produce "weak" solutions and then prove that secretly, these weak solutions are actually smooth. How does one produce weak solutions ? One needs something out of nothing. Such "something out of nothing" theorems are provided by functional analysis for not necessarily the PDE we are trying to solve, but for a slightly different PDE. Then one tries to use the spectral theorem for compact symmetric operators to find a weak solution for the PDE we are trying to solve. This is where compactness kicks in. All of this sounds too abstract, but we will see this in practice soon enough. For now, we will state and prove this theorem (which I promise will be useful later on). This theorem basically says "If we prove estimates for a sequence of functions in one function space, then a subsequence converges in some other space".

Theorem 2.1. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(1) $H^s \subset H^l$ *if* $l < s$. *(Rellich lemma.)*
(2) $H^s \subset C^a(S^1 \times S^1 \ldots)$ *if* $s \geq [\frac{n}{2}] + a + 1$ *where* C^a *is the space of* C^a *functions with the norm* $\|f\| = \max_{S^1\times S^1\ldots} |f(x)| + \max|Df| + \ldots + \max|D^a f|$. *(Rellich-Kondrachov compactness.)*
(3) *Suppose* U *is a bounded domain in* $\mathbb{R}^n$, *then* $C^{k,\alpha}(\bar{U}) \subset C^{k,\beta}(\bar{U})$ *if* $\beta < \alpha$ *and* $C^k \subset C^l$ *if* $l < k$. *(The Hölder space* $C^{k,\alpha}(\bar{U})$ *consists of* $C^{k,\alpha}$ *functions with the norm* $\|f\| = \max\limits_{\bar{U}} |f| + \max|Df| + \ldots + \max|D^k f| + \sum\limits_{|I|=k} \sup\limits_{x\neq y\in\bar{U}} \dfrac{|D^I f(x) - D^I f(y)|}{|x-y|^{\alpha}}$. *This space is a Banach space.)*
(4) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary (in the sense that the boundary is a smooth submanifold of* $\mathbb{R}^n$*). Then,*
   (a) *If* $p < n$, $W^{1,p}$ *is compactly contained in* L^q *where* $q < p^* = \frac{np}{n-p}$ *(please note the strict inequality). The number* p^* *is called a critical exponent.*
   (b) *If* $p > n$, *then* $W^{1,p}$ *is compactly contained in* $C^{0,\gamma}$ *for some* $\gamma > 0$ *determined by* p, n.

Proof. (1) If f_n is a bounded sequence in H^s, then $|\hat{f}_n(\vec{k})|^2(1+|k|^2)^s$ is a bounded sequence of real numbers for all k. Enumerate $\vec{k}$ by positive integers a. Therefore, by completeness of reals, we may assume that there exists a subsequence of functions $a_{1i}(x) = f_{n_i}(x)$ such that $\hat{a}_{1i}(k_1)^2(1+|k_1|^2)^s$ converges to a real number. From this subsequence choose a further subsequence $a_{2i}(x)$ such that $\hat{a}_{2i}(k_1)(1+|k_1|^2)^s$, $\hat{a}_{2i}(k_2)(1+|k_2|^2)^s$ converge. Continue like this. Now choose the diagonal subsequence $b_i(x) = a_{ii}(x)$. It is easy to see that $|\hat{b}_i(\vec{k})|^2(1+|k_i|^2)^s$

is Cauchy for all $\vec{k}$.
Now, $\|b_i - b_j\|^2_{H^l} = \sum_{\vec{k}} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2(1+|k|^2)^l$. When $|k| > N = \epsilon^{1/(l-s)}$, we see that $\sum_{|k|>N} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2(1+|k|^2)^s \frac{1}{(1+|k|^2)^{s-l}} \leq \frac{C}{N^{s-l}} < C\epsilon$. For the other smaller values of $|k|$, choose M is so large that that $b_i(k) - b_j(k)$ is small for all $|k| < N$ and $i, j > M$.

(2) As above, choose the subsequence $b_i(x)$. We will prove that it is Cauchy in the space C^a. If the Fourier series of $b_i - b_j$ (and its derivatives upto order a) converged to it (respectively to its derivatives) uniformly, then,

$$\|b_i - b_j\|_{C^a} = \|\sum \widehat{b_i - b_j}(\vec{k})e^{i\vec{k}.\vec{x}}\|_{C^a} \leq \sum_{p=0}^{p=a} \sum_{\vec{k}} |\widehat{b_i - b_j}(\vec{k})||k|^p \tag{2.1}$$

As before, for $|\vec{k}| > N$, $\sum_{p=0}^{p=a} \sum_{|\vec{k}|>N} |\widehat{b_i - b_j}(\vec{k})||k|^p \leq C\|b_i - b_j\|_{H^s} \sum_{|k|>N} (1+|k|^2)^{a-s} < \epsilon$ for some large N. For $|\vec{k}| \leq N$, as before, we can choose M so that $i, j > M$ implies that the finitely many terms are small.

Now, by the Weierstrass M-test, indeed the Fourier series of $b_i - b_j$ converges uniformly to it (and likewise for its derivatives). So the above argument shows that $\|b_i - b_j\|_{C^a} < \epsilon$ if $i, j > N$.

(3) HW

(4) Omitted. (Evans' book.) The first step is to prove inclusion, and then to prove compactness. Proving inclusion is tricky. (The point is to use the fundamental theorem of calculus and the Hölder inequality in a clever way.)

□

3. Constant-coefficient elliptic operators on the torus

Everything we did earlier holds true for vector-valued periodic functions, i.e., $\vec{u} : S^1 \times \ldots S^1 \to \mathbb{R}^\mu$. (By the way, these things work even when $\mathbb{R}$ is replaced by $\mathbb{C}$ on the right hand side, i.e., for complex-valued functions.) We can define a Fourier series if $\vec{u} \in L^1_{loc}$, $\widehat{\vec{u}(\vec{k})} = \frac{1}{(2\pi)^n} \int\int \ldots \vec{u}(\vec{x})e^{-i\vec{k}.\vec{x}}d^n x$. We can define Sobolev spaces $H^s(S^1 \times S^1 \ldots, \mathbb{R}^\mu)$, and prove the Sobolev embedding and compactness theorems. The Parseval-Plancherel theorem also holds. Moreover, so does the formula relating the Fourier transform of the derivative to that of the function. (By the way, $\langle \vec{u}, \vec{v} \rangle = \sum (1+|k|^2)^s \widehat{\vec{u}}.\widehat{\vec{v}}$.)

Instead of studying $\Delta \vec{u} = \vec{f}$, let us generalise much more. Suppose we want to study

$$L(\vec{u}) = \sum_{|\alpha|=l} [A]_{l,\alpha} D^\alpha \vec{u} + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha} D^\alpha \vec{u} + \ldots = \vec{f},$$

where $A_{k,\alpha}$ are $\mu \times \mu$ matrices of constants, one for each l, α such that $|\alpha| = \alpha_1 + \alpha_2 + \ldots = l$.

Now $L : H^{s+l} \to H^s$ extends uniquely to a bounded linear map (it is easily seen to be bounded linear on the subspace of smooth functions and smooth functions are dense). Take Fourier (series) transform on both sides. Now

$$(\sum_{|\alpha|=l} [A]_{l,\alpha}(ik)^\alpha + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha}(ik)^\alpha + \ldots)\widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k}).$$

This means that for large $|\vec{k}|$, the equation above has a solution if the top order term is invertible, i.e., $\sigma_{\vec{k}} = \sum_{|\alpha|=l} [A]_{l,\alpha}(ik)^{\alpha}$ is an invertible $\mu \times \mu$ matrix for all $|k| \neq 0$. (Note that by homogenity, if it is invertible for all large $|k|$, then it is so for all non-zero ones.) [1]

Definition 3.1. A linear differential operator L with constant coefficients on the torus is said to be elliptic [2] if the principal symbol $\sigma_{\vec{k}}$ is invertible for all $|k| \neq 0$.

[1] Unfortunately, just because the equation has a solution for large $|\vec{k}|$, we cannot conclude that the top-order term is invertible (why not?)

[2] it is called elliptic because for $\mu = 1$ and $n = 2$, we get the equation of an ellipse