NOTES FOR 14 OCT (TUESDAY)

1. Recap

(1) Studied distributions and difference quotients. Sketched a proof of elliptic regularity (largely HW).
(2) Stated

Theorem 1.1. *If $L : H^o \to L^2$ is elliptic, then*
 (a) *$Im(L) \subset L^2$ is closed, and the kernel and cokernel are finite-dimensional.*
 (b) *The kernel consists of smooth functions.*
 (c) *The Cokernel $\simeq ker(L^*) : L^2 \to (H^o)^*$ consists of smooth functions and $Coker(L) \simeq ker(L^*_{form})$.*

(3)

Lemma 1.2. *If U_μ is a coordinate trivialising open cover of M, ρ_μ^2 is a partition-of-unity subordinate to it, and $K_\mu : H^s(S^1 \times S^1 \ldots, \mathbb{R}^r) \to H^s(S^1 \times \ldots, \mathbb{R}^r)$ are compact operators, then $K : H^s(M, E) \to H^s(M, E)$ given by $K(u) = \displaystyle\sum_\mu \rho_\mu K_\mu(\rho_\mu u)$ is also compact where we secretly extend functions supported on U_μ to $S^1 \times S^1 \ldots$ and conversely, functions on $S^1 \times S^1 \ldots$ having support in the image of U_μ are extended by 0 to the manifold.*

2. Elliptic operators-Fredholmness

Proof. (1) This will follow from the construction of parametrices.

Another small observation is that if $TG_1 - I = K_1, G_2T - I = K_2$, then $T + h$ is Fredholm for all h satisfying $\|h\| < \delta$ where δ depends only on upper bounds for $\|G_1\|, \|G_2\|$. Lastly, going over the construction of the parametrices on the Torus, their norms are bounded above depending only on the ellipticity constants and upper bounds on the coefficients.

Cover the manifold with open coordinate trivialising balls of coordinate radius 2 such that the balls of radius 1/2 continue to cover it. Assume that each point of M lies in Q at most such balls (and every ball overlaps with at most Q many balls) of radius 1/2. Now we shall cover each coordinate 1/2 ball by smaller coordinate balls. Here is a lemma:

Lemma 2.1. *Given a ball B of radius $1/2$ centred at the origin in $\mathbb{R}^n$ and a radius r, there exists an open cover by radius-r balls of the closed ball $\tilde{B}$ of radius $\frac{1}{1.9}$ such that every point of $\tilde{B}$ lies in at most 10^{100n^2} balls of the cover. Moreover, every such radius-r ball overlaps with at most 10^{1000n^3} number of balls.*

Proof. Consider a uniform lattice (containing the origin) of size $\frac{r}{2}$ in $\mathbb{R}^n$. Consider open balls of radius-r centred at these lattice points. Every open ball is contained in an open lattice square of size $2r$. Every point in $\mathbb{R}^n$ is contained in at most 10^{10n^2} such squares and hence at most 10^{100n^2} such open balls. Likewise for the overlaps. □

Now we shall choose a fine enough open sets U_μ (basically the open balls from the above lemma), take a partition-of-unity ρ_μ^2, and choose points p_μ. We will decide how many such sets we need later on. Let G_μ be the parametrices from $L^2(S^1 \times S^1 \ldots) \to H^s(S^1 \times S^1, \ldots)$ for $L(p_\mu)$. Note that the norms of G_μ depend solely on the ellipticity constants and upper bounds for the coefficients, and therefore are independent of the size of the open cover U_μ. Now define $G = \sum_\mu \rho_\mu G_\mu \rho_\mu$. Note that $LG(u) : L^2(M,E) \to L^2(M,E)$ as

$$
\begin{aligned}
LG(u) &= \sum_\mu [L, \rho_\mu] G_\mu \rho_\mu u + \rho_\mu (L - L_\mu) G_\mu(\rho_\mu u) + \rho_\mu^2 u + \rho_\mu K_\mu \rho_\mu u \\
&= u + Compact\ u + \sum_\mu \rho_\mu (L - L_\mu) G_\mu(\rho_\mu u) \qquad (2.1)
\end{aligned}
$$

Note that (by Cauchy-Schwarz)

$$
\begin{aligned}
\| \sum_\mu \rho_\mu (L - L_\mu) G_\mu(\rho_\mu u) \|_{L^2} &\leq 10^{10n} Q 10^{1000n^3} \Big(\sum_\mu \|L - L_\mu\|^2_{H^o(U_\mu) \to L^2(U_\mu)} \|G_\mu\|^2_{L^2(S^1 \times \ldots) \to H^o(S^1 \times \ldots)} \|\rho_\mu u\|^2_{L^2} \Big)^{1/2} \\
&\leq \frac{\|u\|_{L^2}}{2} \qquad (2.2)
\end{aligned}
$$

if the cover is chosen fine enough. Hence $I + \sum_\mu \rho_\mu (L - L_\mu) G_\mu \rho_\mu$ is invertible and thus there exists a $\tilde{G}_1$ so that $L\tilde{G}_1 - I = Compact$. This proves that the cokernel (as a subset of L^2) is finite dimensional.

To find a right parametrix $\tilde{G}_2$, i.e., $\tilde{G}_2 L = I + K_2$, we need distributions because it is easier to do it for $\tilde{G}_2 : H^{-o} \to L^2$ rather than $\tilde{G}_2 : L^2 \to H^o$.

Note that L can be extended using distributional derivatives. In fact, if $u \in L^2$ and $v \in H^o$, then $L(u) \in H^{-o}$ and $\langle L(u), v \rangle = (u, L^*_{form} v)_{L^2}$. In other words, it coincides with $L^{*\dagger}_{form} : L^2 \to H^{-o}$. Now G_μ can be easily extended from $H^{-o}(S^1 \times \ldots) \to L^2(S^1 \times \ldots)$ and it is still a parametrix (whose norm is bounded above depending only on the ellipticity constants and upper bounds on the coefficients). The definition $G = \sum \rho_\mu G_\mu \rho_\mu$ by the "HW results" makes sense as a map from $H^{-o}(M,E) \to L^2(M,E)$. We need to then prove that $\| \sum_\mu \rho_\mu G_\mu (L - L_\mu)(\rho_\mu u) \|_{L^2} \leq \frac{1}{2} \|u\|_{L^2}$ to be done. Indeed (by Cauchy-Schwarz),

$$
\begin{aligned}
\| \sum_\mu \rho_\mu G_\mu (L - L_\mu)(\rho_\mu u) \|_{L^2} &\leq \Big(\|G_\mu\|^2_{H^{-o} \to L^2} \|L - L_\mu\|^2_{L^2 \to H^{-o}} \|\rho_\mu u\|^2_{L^2} \Big)^{1/2} \\
&\leq \frac{1}{2} \|u\|_{L^2} \qquad (2.3)
\end{aligned}
$$

for a fine enough cover. This proves that the kernel of L (as a subset of L^2) is finite dimensional. By the regularity theorem for L^2 distributional solutions, every element of the kernel is actually smooth (and is hence in H^o as well). So $L : H^o \to L^2$ has finite dimensional kernel and cokernel. Thus it is Fredholm (and hence its range is closed). This means that if f is smooth, $Lu = f$ has a smooth solution if and only if f is L^2 orthogonal to the kernel of L^* (which by the next two results corresponds to being L^2 orthogonal to the kernel of L^*_{form}).

(2) This follows from elliptic regularity that we proved earlier.

(3) If $u \in ker(L^*)$, then $L^* u(v) = u(Lv) = \langle u, Lv \rangle_{L^2} = 0 \ \forall \ v \in H^o$. Choosing v to be smooth, u is a distributional solution of $L^*_{form} u = 0$. By elliptic regularity u is smooth (the formal adjoint is also elliptic).

□

3. Elliptic operators - Diagonalisability

Suppose L is elliptic (between the space of sections of a real bundle $\Gamma(E)$ and itself (the complex case is not very different)) and symmetric of order $2o$ satisfying Garding's coercivity inequality : $(Lu, u)_{L^2} + \lambda(u, u)_{L^2} \geq \delta(u, u)^2_{H^o}$ (for some positive λ) for all smooth sections. Also assume that $C\|u\|^2_{H^o} \geq B[u, u] = (Lu, u)_{L^2} + \lambda(u, u)_{L^2} \geq \delta\|u\|^2_{H^o}$. Let $u \in H^o$ and $u_n \to u$ in H^o. Since $B[u_n - u_m, u_n - u_m] \leq C\|u_n - u_m\|^2_{H^o}$, we see by completeness of reals that $B[u, u] = \lim_{n\to\infty} B[u_n, u_n]$ exists. We can also prove that $B[v_n, v_n] \to B[u, u]$ if v_n is any other sequence converging to u. Indeed, the following little inequality is crucial for this claim and everything else that follows. Let u, v be smooth sections.

$$B[u, v] = \|u\|_{H^o}\|v\|_{H^o} B[\frac{u}{\|u\|_{H^o}}, \frac{v}{\|u\|_{H^o}}]. \tag{3.1}$$

Now assuming without loss of generality that $\|u\|_{H^o} = \|v\|_{H^o} = 1$, we see by the polarisation identity and the inequalities satisfied by B that $B[u, v] \leq C$. Hence, $B[u, v] \leq C\|u\|\|v\|$.
Using approximation, we can see that the extension of B to H^o is bilinear. Also, $B[u + v, u + v] - B[u, u] - B[v, v] = \lim_{n\to\infty} B[u_n, v_n] + B[v_n, u_n] = 2B[u, v]$ and hence B remains symmetric when extended to H^o. Clearly, B still satisfies the above inequalities.

Since B is symmetric, $B[u, v]$ is a new inner product on H^o which is equivalent to the Sobolev norm and hence Riesz representation implies that for every $f \in L^2$, there is a $u \in H^o$ such that $B[u, v] = (f, v)_{L^2}\ \forall\ v \in H^o$.

Suppose $f \in L^2$ and $B[u, v] = (f, v)_{L^2}\ \forall\ v \in H^o$. Thus, if v is smooth, then $(u, Lv + \lambda v) = (f, v)$. Thus, $u \in H^o$ is a distributional solution to $Lu + \lambda u = f$. Hence it is smooth if f is so.

Define the operator $f \in L^2 \to u_f \in H^o \subset L^2$. This is a compact operator.

Lemma 3.1. *The operator $K(f) = u_f$ is self-adjoint.*

Proof.

$$(v, Kf) = B[Kv, Kf] = B[Kf, Kv] = (f, Kv) \tag{3.2}$$

□

Hence by the spectral theorem, its spectrum consists only of countably many eigenvalues, each eigenspace is finite dimensional, and its eigenvalues are bounded above with 0 as the only accumulation point and its eigenvectors span all of L^2. Moreover, $K - \mu I$ is an isomorphism unless μ is an eigenvalue. Also, by Fredholm theory, $(K - \mu I)u = f$ has a solution if and only if f is orthogonal to the kernel of $K^* - \mu I = K - \mu I$. Using this, here is an exercise :
Exercise : Prove that for L as above, the spectrum of L consists only of eigenvalues (going off to ∞) such that the eigenspaces are finite dimensional and span all of L^2.