NOTES FOR 16 OCT (THURSDAY)

1. Recap

(1) Proved Elliptic operators are Fredholm (with help from Akash!).
(2) Proved that elliptic operators satisfying Garding's inequality and boundedness of a certain bilinear form are diagonalisable.

2. Diagonalisability

In the case of the Hodge Laplacian, in normal coordinates one can easily see that $\Delta_d = \nabla^*\nabla + \textit{lower order terms}$. Thus, $(\Delta_d u, u) = (\nabla u, \nabla u) + (loweru, u)$. Now $|(loweru, u)| \leq C\|\nabla u\|_{L^2}\|u\|_{L^2} \leq \frac{1}{2}\|\nabla u\|_{L^2}^2 + C_1\|u\|_{L^2}^2$. Hence, $(\Delta_d u, u) + (C_1 + \frac{1}{2})(u, u) \geq \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2)$. This proves the Garding coercivity inequality and hence the Hodge theorem. (Actually, all we need is the diagonalisability part because the rest of Hodge follows from the Fredholmness of the elliptic operator Δ_d.)[1] This result opens up a wide area of study (spectral geometry). You can read from Kac's paper "Can you hear the shape of a drum ?"

3. Schauder and $W^{k,p}$ estimates

As in the case of H^s and $C^{k,\alpha}$ we can define the $W^{k,p}(M, E)$ spaces either globally using connections (using either weak derivatives or as the completion of smooth sections) or by partitions-of-unity and the local definition.

If L is elliptic (with smooth coefficients) and $u \in L^p$ is a distributional solution of $Lu = f$ where $f \in W^{k,p}$ (and θ is the order of L), then $u \in W^{k+\theta,p}$ with $\|u\|_{W^{k+\theta,p}} \leq C_{k,p}(\|f\|_{W^{k,p}} + \|u\|_{L^p})$. Likewise, if $f \in C^{k,\alpha}$ and $u \in C^\theta$ is a solution, then $u \in C^{k+\theta,\alpha}$ with $\|u\|_{C^{k+\theta,\alpha}} \leq C_{k,\alpha}(\|f\|_{C^{k,\alpha}} + \|u\|_{C^0})$ (the Schauder estimates). (These things are in L. Nicolaescu's lectures on the geometry of manifolds.) We shall not prove these results. The Schauder estimates are not too hard to prove but the $W^{k,p}$ estimates require some heavy harmonic analysis (the Calderon-Zygmund inequality).

4. Parabolic equations

Let L be an order 2θ elliptic formally self-adjoint operator satisfying the Garding coercivity inequality $(Lv, v)_{L^2} + \lambda(v, v) \geq \delta(v, v)_{H^\theta}$ such that $L : \Gamma(E) \to \Gamma(E)$ and let u_0, f be smooth sections of E. Then the equation $\frac{du}{dt} = -Lu + f,\ u(0) = u_0$ for a section $u : [0, T] \times M \to E$ is called a linear parabolic PDE. The quintessential example of a parabolic PDE is the heat equation $\frac{du}{dt} = \Delta u$. (The equation $\frac{du}{dt} = -\Delta u$ is called the backwards heat equation and is usually badly behaved.)

We typically want u to be smooth on the interior of the parabolic domain and smooth from the right hand side at $t = 0$.

Theorem 4.1. *Every parabolic equation has a unique smooth solution for all time, i.e., on $[0, \infty) \times M$.*

---

[1] Actually, since lower order terms do not make a difference to Fredholmness, this also proves that the above kind of operators plus lower order terms are still Fredholm.

Proof. First we prove uniqueness. Indeed, if there are two solutions, then let $v = u_1 - u_2$. It satisfies $\frac{dv}{dt} = -Lv, v(0) = 0$. Now,

$$\frac{d(v,v)_{L^2}}{dt} = -2(Lv,v) \leq C(v,v)_{L^2}. \tag{4.1}$$

Hence,

$$(v,v)(t) \leq (v,v)(0)e^{-\delta t}.$$

Thus $v \equiv 0$. The estimate on v (an "Energy estimate") is useful in its own right. One can similarly prove that if $\frac{dv}{dt} = -Lv + f$, then $(v,v)(t) \leq C(1+t)$.

Now we prove existence. Let e_n be a countable family of smooth eigenvectors with eigenvalues $\lambda_n \geq 0$ of L spanning L^2. Thus, $u_0 = \sum_n c_n e_n$ for any $u_0 \in L^2$ (and $f = \sum_n f_n e_n$). Since $u_0 \in L^2$, we see that $\sum_n |c_n|^2 < \infty$. First we prove that the quantity $\|u_0\|_k = \sum_n |c_n|^2(1+\lambda_n)^{2k}$ is equivalent to the $H^{k2\theta}$ norm. Indeed, if $\|u_0\|_k < \infty$, then $(u_0, L^k e_n)_{L^2} = \lambda_n^k c_n$. If ϕ is a smooth section, then $\phi = \sum_n \phi_n e_n$. Thus, $L^k\phi \in L^2$ satisfies $(L^k\phi, e_n) = \phi_n\lambda_n^k$. Therefore, $(u_0, L^k\phi) = \sum_n c_n\lambda_n^k\phi_n$ and hence $L^k u_0 = f_k$ in the sense of distributions where $f_k \in L^2$. Therefore, $u_0 \in H^{k\theta}$ and $\|u_0\|_{H^{k2\theta}} \leq C_k\|u_0\|_k$. Conversely, if $u_0 \in H^{2k\theta}$, then $\|L^k u_0\|_{L^2} \leq C\|u_0\|_{H^{2k\theta}} < \infty$. Thus, $(L^k u_0, e_n) = (u_0, L^k e_n) = \lambda_n^k c_n$. Therefore, $\|u_0\|_k < \infty$ and $\|u_0\|_k^2 \leq \tilde{C}_k\|u_0\|_{H^{2k\theta}}^2$.

Define the function $u(t) = \sum_n c_n e^{-\lambda_n t}e_n + \frac{f_n}{\lambda_n}(1 - e^{-\lambda_n t})e_n$. Clearly $u(t) \in L^2$. Moreover, $\|u(t) - u_0\|_{L^2}^2 = \sum_n |c_n|^2(1-e^{-\lambda_n t})^2$ which by DCT converges to 0 as $t \to 0^+$.

Now we proceed to prove that $u(t,x)$ is C^∞ in x for every fixed $t \geq 0$ and that we can differentiate w.r.t x term-by-term. Since $\sum_n c_n e_n$ and $\sum_n f_n e_n$ are smooth (by assumption), their $\|.\|_k$ norms are finite for all (by the equivalence of norms above). Therefore, $\|u\|_{H^{2k\theta}} \leq C_k \ \forall\ k$. Hence u is smooth in x for all fixed $t \geq 0$. Moreover, by Sobolev embedding, the partial sum $s_N(t) = \|\sum_{n=1}^N u_n e_n\|_{C^{k,\alpha}} \leq \tilde{C}_k$ independent of N. Therefore, by Arzela-Ascoli, every subsequence has a subsequence that converges in C^l and in fact the limits are all $u(t)$ because $s_N(t) \to u(t)$ in L^2. Therefore, $u(t) \in C^l$ for all l and the term-by-term derivatives in x converge.

Now note that if $u(t) = \sum_n u_n(t)e_n$ where $\|s_N(t)\|_{H^k} \leq C_k$ independent of $N, t \geq 0$, then $\|s_N(t) - s_N(t_0)\|_k^2 \leq \sum_{n=1}^N (1+\lambda_n)^{2k}(2\lambda_n^2|c_n|^2 + \lambda_n^2|f_n|^2)|t-t_0|^2 \leq \tilde{C}_k\|t-t_0\|^2$ and hence $\|s_N(t) - s_N(t_0)\|_{C^0} < \epsilon$ for $t - t_0$ small (if $t_0 = 0$, then $t \geq 0$). So $u(t,x)$is continuous in (t,x). Actually, this argument shows that $\partial_x^l u(t,x)$ is continuous too.

Likewise, $\|s_N'(t) - s_N'(t_0)\|_{C^0} < \epsilon$ for t close to t_0. So the term-by-term derivatives $s_N'(t)$ converge uniformly to a continuous function $v(t,x)$. Note that $\int_0^t v(a)da = \lim_{N\to\infty}\int_0^t s_N'(a)da = u(s)$ and hence by the FTC, $u'(t,x) = v(t,x)$ and moreover, $u'(t,x)$ is continuous in t, x. (Actually it shows that all the partials in x are also continuous.) Inductively, we can prove that u is smooth on $[0,\infty)\times M$ and that we can differentiate term-by-term.

Finally, an easy calculation shows that u satisfies the equation with the boundary conditions. □

5. Uniformisation theorem

A natural question in Riemannian geometry is the Yamabe problem : Given a compact oriented (M, g_0), find a smooth function $f : M \to \mathbb{R}$ so that $(M, g = e^{-f}g_0)$ has constant scalar curvature. When M is 2-dim, the scalar curvature is upto a factor, the Gaussian curvature K. In such a case, $\int KdA = 2\pi\chi(M)$ (the Gauss-Bonnet theorem) and hence the constant is fixed by the topology of the manifold. So in 2-dim, the equation is strongly linked to the topology of the manifold. This problem is called the Riemannian uniformisation problem.