NOTES FOR 16 SEPT (TUESDAY)

1. Recap

(1) Definition of the Hodge star and the Hodge Laplacian.
(2) Rough Laplacian.

2. Statement of the Hodge theorem and applications

To calculate the De Rham cohomology groups $H^k(M)$, it is useful to have "good, canonical" representatives of each cohomology class. That is, given a class $[\eta]$ consisting of forms $\alpha = \eta + d\gamma$, we want to find the "best" possible $\alpha \in [\eta]$. More precisely, we want to minimise the "energy" $E_g(\alpha) = \int_M |\alpha|_g^2 vol_g$. Suppose α is such a smooth minimiser. Then $\frac{dE(\alpha+td\gamma)}{dt}|_{t=0} = 0 \ \forall \ \gamma$.

$$\frac{dE(\alpha+td\gamma)}{dt}|_{t=0} = \int_M 2\langle d\gamma, \alpha\rangle = 2(d\gamma,\alpha) = 0 \Leftrightarrow (\gamma, d^\dagger\alpha) = 0 \Leftrightarrow d^\dagger \alpha = 0 \tag{2.1}$$

This means that $d\alpha = 0 = d^\dagger\alpha \Rightarrow \Delta_d \alpha = 0$. Conversely, if $\Delta_d\alpha = 0$, then taking inner product with α we see that $\|d\alpha\|^2 + \|d^\dagger\alpha\|^2 = 0$. This means that $d\alpha = 0 = d^\dagger\alpha$. So, ideally, we'd like a statement to the effect of:

Theorem 2.1 (Hodge's theorem)**.** *Suppose (M,g) is compact and oriented. The space of Harmonic forms $\mathcal{H}^k$ among the space of smooth forms is finite dimensional. There is an orthogonal projection H : Smooth k forms $\rightarrow \mathcal{H}^k$ and a unique operator G : Smooth k forms $\rightarrow$ Smooth k forms such that $G(harmonic) = 0$, $Gd = dG, d^\dagger G = Gd^\dagger$ and $I = H + \Delta_d G$. As a consequence, every De Rham cohomology class has a unique harmonic representative. Also, the Hodge Laplacian is diagonalisable, i.e., there is a complete orthonormal basis of eigenvectors (for L^2).*

If we manage to prove this, we have some wonderful conclusions (for compact oriented manifolds):

(1) A weak form of Poincaré duality : The map $H^k(M) \times H^{m-k}(M) \rightarrow \mathbb{R}$ given by $[\omega],[\eta] \rightarrow \int_M [\omega\wedge\eta]$ is non-degenerate. Thus $dim(H^k(M)) = dim(H^{m-k}(M))$. Indeed, choose any metric on M and suppose $\omega \in [\omega]$ is harmonic, i.e., $d\omega = d^\dagger\omega = 0$. Then $*\omega$ is also harmonic because $d*\omega = \pm **d*\omega = 0$ and $d^\dagger *\omega = \pm * d *^2 \omega = \pm * d\omega = 0$. Now $\int \omega \wedge *\omega = \|\omega\|^2 = 0$ if and only if $\omega = 0$, i.e., $[\omega] = [0]$. The Poincaré duality theorem implies that $\chi(M) = dim(H^0(M)) - dim(H^1(M)) + \ldots$ is zero for odd dimensional manifolds. This $\chi(M)$ turns out to be the Euler characteristic, i.e., the alternating sum of the vertices, edges, etc if you triangulate the manifold.

(2) A weak form of the Kunneth formula : $H^k(M\times N) \simeq \oplus_{l=0}^k H^l(M) \otimes H^{k-l}(N)$ with the map being $\oplus[\omega_i]\otimes[\eta_j] \rightarrow \sum[\pi_1^*\omega_i \wedge \pi_2^*\eta_j]$. Choose metrics $g_M, g_N, g_M \times g_N$ on $M, N, M\times N$ respectively, and suppose we represent all classes with their harmonic representatives. Before proceeding further, here is an important lemma.

Lemma 2.2. *Let α be an L^2 form on $M \times N$. Then α can be approximated in L^2 by finite linear combinations of $\pi_1^*\omega \wedge \pi_2^*\eta$ where ω,η are smooth.*

Proof. Let ρ_i, ψ_j be partitions of unity subordinate to coordinate covers $(U_i, \vec{x}_i), (V_j, \vec{y}_j)$ of M, N. Then $\phi_{ij} = \pi_1^* \rho_i \pi_2^* \psi_j$ is a partition of unity on $M \times N$. Now $\alpha = \sum_{i,j} \phi_{ij} \alpha$. Thus wlog α is compactly supported in a coordinate chart $U \times V$. Now $\alpha = \alpha_{IJ} dx^I \wedge dy^J = \rho\psi\alpha + IJdx^I \wedge dy^J$ (where ρ, ψ are bump functions). Choose a sequence of polynomials $\sum c_{n,I,J,P,Q} x^P y^Q$ approximating α_{IJ} uniformly on the support of α. Thus α is approximated in L^2 by finite linear combinations of $\rho x^P dx^I \wedge \psi y^Q dy^J$ which are of the desired decomposable form. □

Then $\pi_1^* \omega_i \wedge \pi_2^* \eta_j$ are harmonic with respect to the product metric. Indeed, obviously they are closed. Now

$$\begin{aligned}(d^{\dagger_{M\times N}} \pi_1^* \omega_i \wedge \pi_2^* \eta_j, \pi_1^* \alpha \wedge \pi_2^* \beta) &= (\pi_1^* \omega_i \wedge \pi_2^* \eta_j, \pi_1^* d\alpha \wedge \pi_2^* \beta \pm \pi_1^* \alpha \wedge \pi_2^* d\beta) \\ &= (\pi_1^* \omega_i, \pi_1^* d\alpha)(\pi_2^* \eta_i, \pi_2^* \beta) + (\pi_1^* \omega_i, \pi_1^* \alpha)(\pi_2^* \eta_i, \pi_2^* d\beta) = 0 \end{aligned} \quad (2.2)$$

This is enough to show that $(d^{\dagger_{M\times N}} \pi_1^* \omega_i \wedge \pi_2^* \eta_j, \alpha) = 0$ for all L^2 α and hence we are done. In fact, one can prove that $\Delta_{M\times N} = \Delta_M + \Delta_N$ (the definition of the RHS is as follows: $\Delta_M \omega$ at the point (p, q) is $\Delta_M i_q^* \omega$ where i_q is the inclusion of $M \times \{q\}$ in $M \times N$). Thus, the map at the level of harmonic forms, the Kunneth map is well-defined. It is clear that it is injective. To prove it is surjective requires some more effort. One has to identify the eigenvectors of the Laplacian and prove it consists of decomposable forms.

Indeed, first one proves that if ω_i, η_j are the orthonormal bases of eigenvectors of Δ_M, Δ_N respectively, then $\pi_1^* \omega_i \wedge \pi_2^* \eta_j$ form a complete orthonormal basis for $L^2(M \times N, g_M \times g_N)$. This can be accomplished by proving that if α is any L^2 form, then $(\alpha, \pi_1^* \omega_i \wedge \pi_2^* \eta_j) = 0$ for all i, j, then α ought to be 0. Indeed, since $\pi_1^* \omega_i \wedge \pi_2^* \eta_j$ approximate $\pi_1 * \omega \wedge \pi_2^* \eta$ in L^2, by the above lemma, we are done.

Second, note that $\Delta_{M\times N}(\pi_1^* \omega_i \wedge \pi_2^* \eta_j) = (\lambda_i + \mu_j) \pi_1^* \omega_i \wedge \pi_2^* \eta_j$. Now if $\Delta u = \lambda u$, then $\|du\|^2 + \|d^\dagger u\|^2 = (u, \Delta u) = \lambda(u, u)$ and hence $\lambda \geq 0$. Now it is easy to see that if ω_i is an eigenvector with eigenvalue λ_i and likewise for η_j and μ_j, then $\pi_1^* \omega_i \wedge \pi_2^* \eta_j$ has $\lambda_i + \mu_j$ as the eigenvalue. If $\lambda_i + \mu_j > 0$, and if η is harmonic, then $(\lambda_i + \mu_j)(\eta, \pi_1^* \omega_i \wedge \pi_2^* \eta_j) = (\eta, \Delta \pi_1^* \omega_i \wedge \pi_2^* \eta_j) = (\Delta \eta, \pi_1^* \omega_i \wedge \pi_2^* \eta_j) = 0$. Therefore, the harmonic forms of $\Delta_{M\times N}$ are finite linear combinations of $\pi_1^* \omega_i \wedge \pi_2^* \eta_j$ where ω_i, η_j are harmonic.

For the flat torus, since we already proved that Δ_d is a constant coefficient symmetric elliptic operator, and that elliptic operators are Fredholm, we see that $\Delta_d \eta = \omega$ can be solved for η if and only if ω is orthogonal to the space of harmonic forms (which we proved is finite dimensional). Moreover, we can choose η to be the unique one having the smallest L^2-norm. So we have $\eta = G(\omega)$. Thus, every form ω can be uniquely written as $H(\omega) + \Delta G(\omega)$. Now $\Delta_d d(G\omega) = dd^\dagger dG\omega = d\Delta_d G\omega = d\omega$. This does not yet show that $d(G\omega) = G(d\omega)$. We need to show that $d(G\omega)$ has the smallest L^2-norm among all such solutions, i.e., it is orthogonal to harmonic forms. But indeed, $(d(G\omega), \alpha) = (G\omega, d^\dagger \alpha) = 0$. Likewise, G commutes with $d^\dagger$. As for the completeness of the eigenfunctions, one can explicitly calculate these eigenfunctions as simply being of the form e^{ikx}. We know that the Fourier functions are complete in L^2 (Parseval-Plancherel).

Seeing how useful this Hodge theorem is, we want to prove it for general compact oriented (M, g). There are several approaches to this. One is to prove such a result for general elliptic operators. (However, that approach has the disadvantage that it does not say much about eigenvalues. So we have to deal with that issue.)