## NOTES FOR 19 AUG (TUESDAY)

### 1. Recap

(1) Proved Sobolev embedding and compactness
(2) Defined elliptic operators.

### 2. Constant-coefficient elliptic operators on the torus

**Definition 2.1.** A linear differential operator $L$ with constant coefficients on the torus is said to be elliptic [1] if the principal symbol $\sigma_{\vec{k}}$ is invertible for all $|k| \neq 0$.

Assume that $L$ is elliptic. Because the $A$ are constants, there exist constants (called the ellipticity constants) $\delta_1, \delta_2$ such that $\delta_2 \|\vec{k}\|^l \|\vec{v}\| \geq \|[\sigma_{\vec{k}}][\vec{v}]\| \geq \delta_1 \|\vec{k}\|^l \|\vec{v}\|$ for all $\mu \times 1$ column vectors $\vec{v}$.

Even for elliptic operators, the above equation for Fourier coefficients cannot always be inverted. However, for sufficiently large $|k|$, it can be inverted to produce an "approximate" solution $\vec{u}_{app}$ whose Fourier coefficients are 0 for $|k| \leq N$ and $\widehat{\vec{u}_{app}}(\vec{k}) = \hat{L}_{\vec{k}}^{-1} \widehat{\vec{f}}(\vec{k})$. We claim that

**Theorem 2.2.** *If $\vec{f}$ is in $H^s$ and $L$ is elliptic, then*

(1) *$\vec{u}_{app}$ is in $H^{s+l}$.*
(2) *The map $G : H^s \to H^{s+l}$ given by $G(f) = \vec{u}_{app}$ is a bounded linear map (with the bound depending on the ellipticity constants, $s, l$, and coefficients of the lower order terms).*
(3) *$L \circ G - I : H^s \to H^s$ and $G \circ L - I : H^{s+l} \to H^{s+l}$ are compact operators. (In simple english, $G$ is an "almost" inverse of $L$. It is called a parametrix for $L$.)*
(4) *If $\vec{u} \in H^{s+l}$ satisfies $L(\vec{u}) = \vec{f}$, then $\|u\|_{H^{s+l}} \leq C(\|f\|_{H^s} + \|u|_{L^2})$ where $C$ depends only on the ellipticity constants, $s, l$, and bounds on the other coefficients (the lower order terms).*

*Proof.* (1) Note that $|\widehat{\vec{u}_{app}}(\vec{k})| \leq C \frac{\|\widehat{\vec{f}}(\vec{k})\|}{\|\vec{k}\|^l}$ if $|\vec{k}| \geq N$ where $N$ is sufficiently (depending on the ellipticity constants and the coefficients of the lower order terms) large. Indeed, the magnitude of the lower order terms is less than $C(\|\vec{k}\|^{l-1} + \|\vec{k}\|^{l-2} + \ldots \leq C\|\vec{k}\|^{l-1})$ if $\|\vec{k}\| > 1$. Now $\|[\sigma_{\vec{k}} + lower][\vec{v}]\| \geq (\delta_1 \|\vec{k}\|^l - C\|\vec{k}\|^{l-1})\|\vec{v}\|$. Of course if $|\vec{k}| \geq N$ is large, then $\|\hat{L}[\vec{v}]\| \geq c\|\vec{v}\|$ where $c > 0$. Hence $\|\hat{L}^{-1}[\vec{v}]\| \leq C\|\vec{k}\|^{-l}\|\vec{v}\|$ for large $N$.

The above easily implies that $\vec{u}_{app} \in H^{s+l}$. Moreover, $\|\vec{u}_{app}\|_{H^{s+l}} \leq C\|f\|_{H^s}$.

(2) The last inequality implies this result.

(3) $K(f) = L \circ G(f) - f = L(u_{app}) - f = -\sum_{|k|<N} \widehat{\vec{f}}(\vec{k}) e^{i\vec{k}.\vec{x}}$. Now $K(f)$ is smooth and is hence in $H^a$ $\forall$ $a$. By the Rellich compactness lemma, $K(f) : H^s \to H^s$ is compact. Now $G(L(u)) - u = -\sum_{|k|<N} \hat{u}(k) e^{i\vec{k}.\vec{x}}$. As before this is a smooth function and hence by the Rellich lemma, $G \circ L - I$ is compact.

[1] it is called elliptic because for $\mu = 1$ and $n = 2$, we get the equation of an ellipse

(4) Taking Fourier series on both sides, $\hat{L}\widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k})$. Of course, for large $|k|$, $u$ coincides with $u_{app}$. For small $|k| < N$, $(1+|k|)^{s+l} \leq (1+N)^{s+l} \leq C$ where $C$ depends only on $N, s, l$ and hence only on the ellipticity constants, $s, l$, and the bounds on the lower order coefficients. This proves the result.

□

Now we define a useful notion in functional analysis.

**Definition 2.3.** Suppose $H_1, H_2$ are Hilbert spaces. A bounded linear operator $T : H_1 \to H_2$ is called Fredholm if $ker(T), Coker(T)$ are finite-dimensional.

We prove the following useful theorem about Fredholm operators. In these results, we use the easy fact that if $T$ is a bounded linear operator and $K$ is compact, then $T \circ K$ and $K \circ T$ are compact. We also use a slightly more difficult fact that if $K$ is compact, then $K^*$ is so as well.

**Theorem 2.4.** *Let $T : H_1 \to H_2$ be a bounded linear operator.*

(1) *If $Im(T)$ is closed, then $Coker(T)$ is naturally a Banach space isomorphic to $Im(T)^{\perp}$. Therefore, $Coker(T)$ is a Hilbert space.*
(2) *If the range of $T$ is closed, then $Coker(T)^* \simeq Ker(T^*)$ where $T^* : H_2^* \to H_1^*$.*
(3) *If $Coker(T)$ is finite dimensional, then the range is closed.*
(4) *$T$ is Fredholm if and only if $T^*$ is so.*
(5) *$T$ is Fredholm if and only if there exist bounded linear maps $G_1, G_2 : H_2 \to H_1$, such that $G_1 \circ T - I$, $T \circ G_2 - I$ are compact operators.*
(6) *The set of Fredholm operators $S \subset B(H_1, H_2)$ is open.*
(7) *Suppose $I \subset \mathbb{R}$ is a connected set. If $F(t) : I \subset \mathbb{R} \to S$ is a continuous map, then the index $Ind(F(t)) = dim(Ker(F(t))) - dim(Coker(F(t)))$ is a constant.*
(8) *If $K : H_1 \to H_2$ is a compact operator and $T$ is Fredholm, then $T + K$ is Fredholm with the same index.*

*Proof.* (1) Define $\|[y]\| = \inf_{y\in[y]} \|y\|$. By Riesz's lemma, the infimum is attained as a minimum $y_0 \in Im(T)^{\perp}$. The map $[y] \to y_0$ is linear and an isomorphism. We are done.

(2) Take $\rho \in ker(T^*) \subset H_2^*$ to $\lambda \in Coker(T)^*$ where $\lambda([y]) = \rho(y)$. This map $V : ker(T)^* \to Coker(T)^*$ is well-defined because $\rho(Tx) = T^*(\rho)(x) = 0$ by definition. It is clearly a linear map (and bounded). If the range is closed, then $Coker(T)$ is a Hilbert space isomorphic to $Im(T)^{\perp}$. Consider the map $U : Coker(T)^* \to H_2^*$ given by $U(\lambda)(v) = \lambda([v])$. This map is clearly linear and bounded. It can be easily seen to invert $V$.

(3) Let $X = Ker(T)^{\perp}$ and let $v_1, \ldots, v_n \in H_2$ be such that $[v]_i$ form a basis for $Coker(T)$. Denote by $C$ the span of $v_i$ in $H_2$. Note that $C \cap Im(T) = \{0\}$. Define a map $S : X \oplus C \to H_2$ as $S(x, c) = T(x) + c$. This map is clearly $1-1$. It is onto because $[y] = \sum_i c_i[v_i]$ and hence $y = \sum_i c_i v_i + T(x)$. Thus, $S$ is a bounded linear isomorphism. Therefore, by the open mapping theorem, it is a homeomorphism. Hence, $S(x, 0) = Im(T)$ is closed.

(4) Suppose $T$ is Fredholm. Since $Coker(T)$ is f.d., the range is closed. Thus, $ker(T^*) = Coker(T)^*$ which is f.d. Moreover, $Coker(T*) = Im(T*)^{\perp} = Ker(T)$ (why?) and hence f.d.

(5) cont'd....

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