NOTES FOR 21 AUG (THURSDAY)

1. Recap

(1) Proved properties of the parametrix and defined Fredholm operators.
(2) Proved a few properties of Fredholm operators.

2. Constant-coefficient elliptic operators on the torus

Theorem 2.1. (1) *If the range of T is closed, then $Coker(T)^* \simeq Ker(T^*)$ where $T^* : H_2^* \to H_1^*$.*
(2) *If $Coker(T)$ is finite dimensional, then the range is closed.*
(3) *T is Fredholm if and only if T^* is so.*
(4) *T is Fredholm if and only if there exist bounded linear maps $G_1, G_2 : H_2 \to H_1$, such that $G_1 \circ T - I$, $T \circ G_2 - I$ are compact operators.*
(5) *The set of Fredholm operators $S \subset B(H_1, H_2)$ is open.*
(6) *Suppose $I \subset \mathbb{R}$ is a connected set. If $F(t) : I \subset \mathbb{R} \to S$ is a continuous map, then the index $Ind(F(t)) = dim(Ker(F(t))) - dim(Coker(F(t)))$ is a constant.*
(7) *If $K : H_1 \to H_2$ is a compact operator and T is Fredholm, then $T + K$ is Fredholm with the same index.*

Proof. (1) Done.
(2) Done.
(3) Done
(4) If T is Fredholm, then $T : ker(T) \oplus ker(T)^{\perp} \to Coker(T) \oplus Im(T)$ is bounded linear and defines an injective map $T_1 : ker(T)^{\perp} \to H_2$. Define $G(a \oplus b) = T_1^{-1}(b)$. Clearly, $G \circ T - I$ is a projection onto a finite dimensional subspace and hence compact. Now $T \circ G(a \oplus b) - a \oplus b = T(T_1^{-1}(b)) - a \oplus b = -a \oplus 0$ which is another projection and hence compact.
Conversely, if there exists such G_1, G_2, then $G_1 T = I + K$. Therefore $Ker(T) \subset Ker(G_1 T) = Ker(I + K)$ which we claim is finite-dimensional. Indeed, if v_i is a bounded sequence in $Ker(I + K)$, then $Kv_i = -v_i$ has a convergent subsequence. But the unit ball is compact in a Banach space if and only if the space is finite-dimensional (Riesz's lemma). Thus $ker(T)$ is finite dimensional. On the other hand, we know that for compact operators, $I + \tilde{K} = TG_2$ has closed range. Therefore, $Coker(TG_2)^* \simeq ker(G_2^* T^*) = Ker(I + \tilde{K}^*)$. Hence, $dim(Coker(TG_2))$ is finite. Now, take the map $[v] \to [v]$ from $Coker(TG_2)$ to $Coker(T)$. Its is clearly well-defined, linear, and onto. Thus, $dim(Coker(T)) < \infty$.
(5) If F is Fredholm, there exists a G so that $FG = I + K_1$ and $GF = I + K_2$. Now if F were invertible, then $(F + p)^{-1} = F^{-1}(1 + F^{-1}p)^{-1} = F^{-1} \sum (-1)^i (F^{-1}p)^i$ which makes sense if $\|p\|$ is small. Now, define $G_p = G(1 + Gp)^{-1}$ for small p. Now $(F + p)G_p = FG(I + Gp)^{-1} + pG(I + Gp)^{-1} = (I + Gp)^{-1} + K_1(I + Gp)^{-1} + pG(I + Gp)^{-1} = H_p + K$ where $H = (I + Gp)^{-1} + pG(I + Gp)^{-1}$. Clearly when p is small, then H_p is invertible.
Thus $(F + p)G_p = (I + KH_p^{-1})H_p$. Now define $\tilde{G}_p = G_p H_p^{-1}$. So $(F + p)\tilde{G}_p = I + compact$. Likewise we can find another $\tilde{G'}_p$ which is an approximate left inverse for small p. Thus $F + p$ is Fredholm for all small p if F is so.

(6) If we prove that $Ind(F+p) = Ind(F)$ for all small p, we will be done because I is connected. First we prove that for small p, there is a linear transformation $A_p : Ker(T) \to Coker(T)$ so that $Ker(T+p) = Ker(A_p)$ and $Coker(T+p) = Coker(A_p)$. For operators between finite-dimensional spaces, the index equals the difference in dimensions and is hence a constant. Indeed, writing $T : Ker(T)^{\perp} \oplus Ker(T) \to Im(T) \oplus Coker(T)$ as $T = \begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}$ where T' is an isomorphism. Write $p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Take $A_p = -c(T'+a)^{-1}b + d$. It can be verified that A_p does the job.

(7) If $G_1 T = I + K_1$ and $TG_2 = I + K_2$, then $G_1(T+K) = I + K_1 + G_1 K = I + compact$ and likewise. Thus $T+K$ is Fredholm. Now $T + sK$ has locally constant index where $s \in [0,1]$. Hence $Ind(T+K) = Ind(T)$.

□

We define the formal adjoint L^*_{form} of L as follows.

Definition 2.2. If $Lu = \sum_{\alpha,p} [A]_{p,\alpha} D^{\alpha} u$, then define the formal adjoint $L^*_{form} v = \sum_{\alpha,p} [A^*]_{p,\alpha} (-1)^{|\alpha|} D^{\alpha} v$. It satisfies $\langle Lu, v\rangle_{L^2(S^1 \times S^1 \ldots)} = \langle u, L^*_{form} v\rangle_{L^2(S^1 \times S^1 \ldots}$ whenever u, v are smooth functions.

We have the following easy lemma.

Lemma 2.3. *If L is elliptic, then so is L^*_{form}.*

Using the above theorems and some more work we conclude the following.

Theorem 2.4. *If L is elliptic, then*

(1) *$Im(L) \subset H^s$ is closed, and $ker(L) \subset H^{s+l}$ and $coker(L) = \frac{H^s}{Im(L)}$ are finite-dimensional subspaces. (Fredholm's alternative.)*

(2) *$Ker(L)$ consists of smooth functions.*

(3) *Suppose $L : H^l \to L^2$. Then $Coker(L)^* \simeq Ker(L^* : L^2 \to (H^l)^*)$ consists of smooth functions and $Coker(L) \simeq Ker(L^*_{form})$.*

(4) *If f is in H^s and $u \in L^2$ is a distributional solution of $Lu = f$, then u is in H^{s+l}. (Elliptic regularity.)*

Proof. (1) By the above theorems, since there is a parametrix for elliptic operators, $L : H^{s+l} \to H^s$ is Fredholm. Hence its kernel and cokernel are finite dimensional and its range is closed.

(2) This follows from the last result in this lemma.

(3) If $u \in (L^2)^* \cap ker(L^*)$, then $L^* u(v) = u(Lv) = \langle u, Lv\rangle_{L^2} = 0$ for all $v \in H^l$. Thus, choosing v to be a smooth function, we see that u is a distributional solution to $L^*_{form} u = 0$. Since the formal adjoint is also elliptic, by the previous part, its kernel consists of smooth functions. Thus, $Coker(L) \simeq Ker(L^*) \subset Ker(L^*_{form})$. If $u \in Ker(L^*_{form})$, then $L^* u(v) = \langle u, Lv\rangle = \langle L^*_{form} u, v\rangle = 0$ for all smooth v. By approximation of H^l functions using smooth functions, we see that it holds for all $v \in H^l$ and hence $u \in Ker(L^*)$. Thus we are done.

(4) Suppose $\phi : S^1 \times S^1 \ldots$ is any smooth function. Since $u \in L^2$ is supposedly a distributional solution (by the way u need not be in L^2 for this to be true, it need be only a distribution), $\langle L^*_{form} \phi, u\rangle_{L^2} = \langle \phi, f\rangle_{L^2}$. This means that (by the Parseval-Plancherel theorem), $\sum_{\vec{k}} \hat{\phi}^T \overline{\widehat{L}\hat{u}} = \hat{\phi}^T \bar{\hat{f}}$ for all ϕ. Now choose ϕ to have Fourier series such that $\hat{\phi}(k) = 1$ if and only if $\vec{k} = \vec{a}$

and 0 otherwise. Then $\widehat{L}(\vec{a})\hat{u}(\vec{a}) = \hat{f}(\vec{a})\ \forall\ \vec{a}$. This observation implies that $\hat{u} = \hat{u}_{app}$ for all $|k| \geq N$. Hence, by the previous results, $u \in H^{s+l}$.

□

Remark 2.5. The above implies that elliptic operators with constant coefficients on the torus are Fredholm operators between Sobolev spaces. So their index is constant under small (arbitrary) perturbations and under compact perturbations. This index turns out to be given by an integral over the torus of some differential form (whose De Rham cohomology class depends only on the principal symbol of L). This is a special case of the Atiyah-Singer index theorem which deals with general elliptic operators on general manifolds.

3. Riemannian manifolds and metrics on vector bundles

In order to define $\Delta u = f$ on a manifold, unfortunately, we cannot do this locally by choosing coordinates and saying $\sum_i \frac{\partial^2}{\partial(x^i)^2} u = f$ because if we change coordinates, then the PDE will not be the same. So how can hope to even set up the Poisson PDE on a manifold ?

Another way of looking at the Laplacian is $\Delta = \nabla . \nabla$. So if we can define a dot product on every tangent space, and define the ∇ operation, then we can define the Laplacian. Why would we care about defining the Laplacian ? Among other things, it gives insight into the De Rham cohomology of the manifold.