NOTES FOR 21 OCT (TUESDAY)

1. Recap

(1) Proved existence and uniqueness for a parabolic equation (by writing an "explicit" solution via eigenfunctions and proving one can differentiate term by term).
(2) Motivated the Riemannian uniformisation problem (special case of Yamabe).

2. The Riemannian uniformisation theorem using the method of continuity

Writing the resulting PDE down, we get (See list of formulas in Riemannian geometry on wikipedia to get the correct formula),

$$\Delta f = Ke^{-f} - K_0, \tag{2.1}$$

where K_0 is the Gaussian curvature of g_0, K is the new curvature, Δf is locally, at a point where we choose coordinates such that $g_0(p)$ is the Euclidean metric up to the second order, $\Delta f(p) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. The question is - Can we solve this equation ? If so, is the solution unique ? The answer (which is supposedly blowing in the wind) is provided by the Riemannian uniformisation theorem -

Theorem 2.1. *In every conformal class of metrics $[g]$ on a compact oriented surface, there exists a unique (up to rescalings by positive constants) metric of constant curvature.*

Before we prove the theorem (we won't actually prove it for $g = 0$), firstly for $g \geq 2$, by solving the linear PDE $\Delta f + K_0 = 2\pi\chi(M)$, we can assume WLog that $K_0 < 0$ everywhere.

Proof. It is actually quite hard to prove (shockingly enough) this theorem for genus $g = 0$, i.e., for a sphere ! (Of course there is one metric of positive constant curvature that even children (who do not believe the flat-earth theory) know about. The issue is that are there other conformal classes ? (There aren't) If there are, how do you prove that they have such metrics ? The technique we are going to describe below will run into serious challenges for $g = 0$.) In fact, this is no coincidence. It turns out that one generalisation of this observation has been proven recently by Chen-Donaldson-Sun (and apparently independently by Tian). It is called the Yau-Tian-Donaldson conjecture. Another generalisation called the Yamabe problem was solved earlier.
Let us take the next case of $g = 1$. Note that by the Gauss-Bonnet theorem, $\int KdA = \int_M K_0 dA_0 = 2\pi(2-2g) = 0$. Therefore we want $K = 0$. This means we have to solve

$$\Delta f = -K_0.$$

Let's prove uniqueness first. Indeed, if f_1, f_2 are two solutions, then $\Delta(f_1 - f_2) = 0$. Multiplying by $f_1 - f_2$ and integrating-by-parts we get

$$-\int_M |\nabla(f_1 - f_2)|^2 = 0.$$

Therefore $f_1 - f_2$ is a constant. The point of this calculation is "If you want to prove that the kernel of some operator is trivial, multiply by something and integrate-by-parts". Since the Laplacian is elliptic, the Fredholm alternative shows that we are done for $g = 1$.

For higher genus, we want $K < 0$. Now we are faced with a nonlinear PDE. Here is a beautiful method (originally due to Bernstein) to handle such PDE. It is called the method of continuity. Consider the following family of PDE indexed by a number $0 \leq t \leq 1$.

$$\Delta f_t = -e^{-f_t} - tK_0 + (1-t). \tag{2.2}$$

At $t = 0$ there is obviously a solution $\phi_0 = 0$. If we prove that the set of t for which there exists a smooth solution is both open and closed, then by connectedness, the set is $[0, 1]$.

(1) Openness : Basically, given a solution at $t = t_0$, we need to prove that there are solutions nearby. Consider the following map,

$$T(t, f) = \Delta f + e^{-f} + tK_0 - (1-t). \tag{2.3}$$

Naively speaking, if this was a map between finite dimensional things, then by implicit function theorem, if its derivative with respect to f is surjective then we will be done. Indeed, there is an implicit function theorem on Banach spaces.

Theorem 2.2. *Suppose X, Y, Z are Banach spaces, $C \subset X \times Y$ is open, and $f : C \to Z$ is C^1. Suppose $(a, b) \in C$ and $v \to Df_{(a,b)}(0, v)$ is a Banach space isomorphism from Y onto Z. Then locally, $z = f(x, y)$ can be solved for to yield a C^1 function g such that $y = g(x, z)$.*

Remark 2.3. In fact, there is one for Banach manifolds, i.e., Hausdorff topological spaces equipped with a maximal atlas consisting of open sets isomorphic to a open subsets of Banach spaces with the transition functions being smooth, i.e., the Fréchet derivatives (in the usual sense) exist as multilinear bounded maps. Typically they are required to be separable and metrisable. In fact a theorem of Henderson states that such beasts are diffeomorphic to open subsets of the separable Hilbert space.

Remark 2.4. The theorem follows from the inverse function theorem on Banach spaces, whose proof is exactly word-to-word the same as the finite-dimensional one. (The contraction mapping principle.) You may look at Lang's book for it.

The appropriate Banach spaces to consider are $\mathbb{R} \times \mathbb{C}^{k+2,\alpha}$ and $\mathbb{C}^{k,\alpha}$ for a given integers $k \geq 0$ and $\alpha > 0$. Why the α? (Hölder space) It is for a technical reason as we shall see in a moment. The "derivative" with respect to f being surjective is the same as saying, for every $v \in \mathbb{C}^{k,\alpha}$ there exists a $u \int C^{k+2,\alpha}$ such that

$$\begin{aligned} &\frac{d}{ds}|_{s=0} T(t_0, f_{t_0} + su) = v \\ &\Rightarrow \Delta u - e^{-f_{t_0}} u = v. \end{aligned} \tag{2.4}$$

Borrowing from our intuition from linear algebra (it is easy to verify that the above equation for u is self-adjoint), i.e., using the Fredholm alternative, we simply need to show that the kernel is trivial. Indeed if u is in the kernel, then

$$\begin{aligned} &\Delta u = e^{-f_{t_0}} u \\ &\Rightarrow -\int_M |\nabla u|^2 dA_0 = \int_M u^2 e^{-f_{t_0}} dA_0. \end{aligned}$$

which means that $u = 0$.

(2) Closedness : This is usually the harder part of any method of continuity. What does it mean for the set to be closed ? It means that for any sequence $t_n \to t$ such that f_{t_n} exist, there exists a solution f_t at t. In other words, if we can prove that a subsequence $f_{t_{n_k}} \to f_t$ in $C^{2,\alpha}$ then we will be done. Beautifully enough, the Arzela-Ascoli theorem implies that (do this as an exercise) if $\beta > \alpha$ and a sequence w_n is bounded independent of n in $C^{2,\beta}$, then a subsequence converges in $C^{2,\alpha}$! Thus, to show closedness, it is enough to prove that solutions to equation 2.2 have a uniform $C^{2,\beta}$ estimate independent of t.

Indeed, such estimates are proven by improving upon lower order estimates -

Let's see if we can at least prove that $\|f_t\|_{C^0} \leq C$. Note that at the minimum of f_t, $\Delta f_t \geq 0$ and hence $f_t \geq -C$ (looking at the PDE). Since $K_0 \leq 0$, then a similar argument at the maximum will show $f_{max} \leq C$.

Actually, now we have some standard results in PDE theory (read Kazdan's notes for instance) that say effectively the following : If the right hand side of $\Delta f = h$ is bounded in L^p for all large p, then f is actually bounded in $C^{1,\alpha}$ for some $\alpha > 0$ (It is basically L^p regularity + Sobolev embedding). There is another result (Schauder's estimates) that implies that if the right hand side of $\Delta f = h$ is bounded in $C^{0,\alpha}$ and $\|f\|_{C^0} \leq C$, then actually $\|f\|_{C^{2,\alpha}} \leq C$. So combining all of these, we get our desired estimates. (These are called "a priori" estimates.)

As for uniqueness, suppose f_1, f_2 satisfy the equation for $K < 0$. Then

$$\Delta(f_1 - f_2) = K(e^{-f_1} - e^{-f_2})$$

$$\Rightarrow -\int_M |\nabla(f_1 - f_2)|^2 = K \int_M (f_1 - f_2)(e^{-f_1} - e^{-f_2}). \tag{2.5}$$

This means that $f_1 - f_2$ is a constant.

For $K = 0$ it is easy but for $K > 0$ it is much harder.

By the way, for $K > 0$, here is a way to prove some things : Firstly, in the conformal class of the usual round metric, there exists a constant curvature metric (the round one). Then assuming one knows complex geometry one proves that there is only one complex structure on the sphere. (This involves a little bit of algebraic geometry.) Thus there is only one conformal class and we are done.

□

3. Killing-Hopf theorem

So what if we find a constant Gaussian curvature metric on a surface ? Big deal! Actually, there is an old theorem called the Killing-Hopf theorem that implies (in the special case of surfaces) that a constantly curved surface is isometric to a quotient of one of the following :

(1) $\mathbb{R}^2$ with the Euclidean metric. (Flat earth according to some idiots.)
(2) S^2 with the standard round metric.
(3) $\mathbb{H}^2$, i.e. the upper half-plane with the metric $g = \frac{dx^2+dy^2}{y^2}$.

In other words,
Given a conformal class of metrics $[g]$ on a compact oriented surface, there is a unique representative in the conformal class of unit volume such that it is isometric to a quotient of one of the things above. In other words, if you care only about measuring angles (and not distances), you are always

a quotient of the standard spaces. Already it looks like complex analysis might play a role here. (Recall that a biholomorphism preserves angles, and vice-versa in the complex plane.)

4. Complex manifolds

Seeing that we are quickly entering complex analysis, let us define complex manifolds. A complex manifold M of dimension n is a smooth manifold of dimension $2n$ such that it is locally diffeomorphic to an open set of C^n and such that the transition maps are biholomorphisms.
Hold on! What is the meaning of a holomorphic map from C^n to C ? It is simply holomorphic on each of the coordinates, i.e., the complex partial derivatives exist.

Anyway, the simplest complex manifolds are those of dimension 1. They are called Riemann surfaces. What are examples of Riemann surfaces ? Well $\mathbb{C}$ is one. Any quotient of $\mathbb{C}$ by a lattice, i.e., a torus $\frac{\mathbb{C}}{\mathbb{Z}^2}$ is one. A famous one is $\mathbb{CP}^1$, i.e., $\mathbb{C}^2 - \{(0,0)\}$ quotiented out by : (X_0, X_1) identified with $\lambda(X_0, X_1)$ where $\lambda \neq 0$ is complex. It is not hard to see that this is the same as the sphere. The upper half-plane is one such example. Actually, it is a nice exercise to prove that any isometry of the upper half-plane is actually a biholomorphism. Another nice exericse is to show that every complex manifold is orientable. (Just calculate the Jacobian and you will see....) A third nice exercise is to show that if you quotient out a complex manifold by a group of biholomorphisms in such a way that the quotient is a smooth manifold, then the quotient is also a complex manifold.

So, the Riemannian uniformisation theorem implies that given a conformal class of metrics $[g]$ on a compact oriented surface M of genus ≥ 1, one can treat $(M, [g])$ as a *Riemann surface* by simply treating the manifold as a quotient of either the plane or the upper half-plane. The real question is, does every compact Riemann surface of genus ≥ 1 arise this way and is every genus 0 compact Riemann surface always $\mathbb{CP}^1$?

5. The uniformisation theorem for Riemann surfaces

The answer to the previous question is yes. Indeed,

Theorem 5.1. *Every Riemann surface (even the noncompact ones) arises as a quotient of the plane, the upper half-plane, or* $\mathbb{CP}^1$.

We can prove this for compact Riemann surfaces using the Riemannian uniformisation theorem. All we need to so is somehow relate conformal classes of metrics to complex structures. To do this, we need to understand a simple question :
In what way can we relate $\mathbb{C}$ and $\mathbb{R}^2$?
This question sounds silly, but what we mean is the following - If I give you $\mathbb{R}^2$, what information would you need to call it $\mathbb{C}$? You would have to somehow make it a complex vector space. So you would need to know what multiplication of (a, b) with $\sqrt{-1}$ means. Indeed, usually, $z = x + \sqrt{-1}y$ and therefore $\sqrt{-1}z = \sqrt{-1}x - y.=$, i.e., $\sqrt{-1}(1, 0) = (0, 1)$ and $\sqrt{-1}(0, 1) = (-1, 0)$. So multiplication by $\sqrt{-1}$ is simply a linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that $J^2 = -I$. (Exercise : Prove that if $J : V \to V$ where V is a real vector space is such that $J^2 = -I$, then indeed V is $2n$ dimensional and that there is a real basis $e_1, w_1, e_2, w_2, \ldots$ of V such that $Je_i = w_j$, $Jw_i = -e_i$.) This is known as an almost complex structure.

So naively speaking, if we have a linear map J from the tangent bundle of a surface M to itself such that $J^2 = -I$, then we can treat M as a complex manifold such that locally indeed there exist coordinates (x, y) so that $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ and $J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$? The answer is yes. But in higher dimensions, it is NO. (There is an additional condition called Integrability. It is a deep theorem called the Newlander-Nirenberg theorem.)

Anyway, what does all of this have to do with what we were discussing ? If you give me a conformal class of metrics, I can come up with an almost complex structure. Indeed, J is simply "rotate "anticlockwise" (with respect to the given orientation) by ninety degrees". Likewise, if you give me a Riemann surface, then here is a conformal class of metrics : Choose any Hermitian metric h on the complex tangent bundle (spanned locally by $\frac{\partial}{\partial z}$). This defines a Riemannian metric if you identify the complex tangent bundle with the real one as above. You can even prove that an isometry of a metric induces a biholomorphism between the corresponding Riemann surfaces. This along with the Riemannian uniformisation theorem proves the Riemann surface uniformisation theorem.